Conjugacy separability of non-positively curved groups

Ashot Minasyan

University of Southampton

Groups and Geometry in the South East
Oxford, 27.05.2016

Background and motivation

Residual finiteness and conjugacy separability are two classical residual properties of groups considered in Combinatorial Group Theory.

Background and motivation

Residual finiteness and conjugacy separability are two classical residual properties of groups considered in Combinatorial Group Theory.

For $x, y \in G$ we will write $x \sim y$ if $\exists\, z \in G$ s.t. $y = zxz^{-1}$.

Residual finiteness and conjugacy separability are two classical residual properties of groups considered in Combinatorial Group Theory.

For $x, y \in G$ we will write $x \sim y$ if $\exists\, z \in G$ s.t. $y = zxz^{-1}$.

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.

Background and motivation

Residual finiteness and conjugacy separability are two classical residual properties of groups considered in Combinatorial Group Theory.

For $x, y \in G$ we will write $x \sim y$ if $\exists\, z \in G$ s.t. $y = zxz^{-1}$.

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

Background and motivation

Residual finiteness and conjugacy separability are two classical residual properties of groups considered in Combinatorial Group Theory.

For $x, y \in G$ we will write $x \sim y$ if $\exists\, z \in G$ s.t. $y = zxz^{-1}$.

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

Equivalently, G is RF $\iff \bigcap_{N \lhd_f G} N = \{1\}$.

Basic properties

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

RF and CS are useful for various purposes:

Basic properties

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

RF and CS are useful for various purposes:

- a f.g. RF group G is Hopfian, i.e., every surjective $\varphi : G \twoheadrightarrow G$ is injective, so $\varphi \in Aut(G)$ (Mal'cev)

Basic properties

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

RF and CS are useful for various purposes:

- a f.g. RF group G is Hopfian, i.e., every surjective $\varphi : G \twoheadrightarrow G$ is injective, so $\varphi \in Aut(G)$ (Mal'cev)
- a f.p. RF group has solvable word problem (Mal'cev)

Basic properties

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

RF and CS are useful for various purposes:

- a f.g. RF group G is Hopfian, i.e., every surjective $\varphi : G \twoheadrightarrow G$ is injective, so $\varphi \in Aut(G)$ (Mal'cev)
- a f.p. RF group has solvable word problem (Mal'cev)
- a f.p. CS group has solvable conjugacy problem (Mostowski)

Basic properties

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

RF and CS are useful for various purposes:

- a f.g. RF group G is Hopfian, i.e., every surjective $\varphi : G \twoheadrightarrow G$ is injective, so $\varphi \in Aut(G)$ (Mal'cev)
- a f.p. RF group has solvable word problem (Mal'cev)
- a f.p. CS group has solvable conjugacy problem (Mostowski)
- if G is f.g. and RF then $Aut(G)$ is also RF (Baumslag)

Basic properties

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

RF and CS are useful for various purposes:

- a f.g. RF group G is Hopfian, i.e., every surjective $\varphi : G \twoheadrightarrow G$ is injective, so $\varphi \in Aut(G)$ (Mal'cev)
- a f.p. RF group has solvable word problem (Mal'cev)
- a f.p. CS group has solvable conjugacy problem (Mostowski)
- if G is f.g. and RF then $Aut(G)$ is also RF (Baumslag)
- if G is f.g., CS and $Aut_{pi}(G) = Inn(G)$ then $Out(G)$ is RF (Grossman)

Basic properties

Definition

- A gp. G is residually finite (RF) if $\forall\, x, y \in G$ with $x \neq y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \neq \psi(y)$.
- A gp. G is conjugacy separable (CS) if $\forall\, x, y \in G$ with $x \not\sim y$, $\exists$ a homom. $\psi : G \to M$, with $|M| < \infty$, s.t. $\psi(x) \not\sim \psi(y)$.

RF and CS are useful for various purposes:

- a f.g. RF group G is Hopfian, i.e., every surjective $\varphi : G \twoheadrightarrow G$ is injective, so $\varphi \in Aut(G)$ (Mal'cev)
- a f.p. RF group has solvable word problem (Mal'cev)
- a f.p. CS group has solvable conjugacy problem (Mostowski)
- if G is f.g. and RF then $Aut(G)$ is also RF (Baumslag)
- if G is f.g., CS and $Aut_{pi}(G) = Inn(G)$ then $Out(G)$ is RF (Grossman)

Recall: $Aut_{pi}(G) = \{\alpha \in Aut(G) \mid \alpha(g) \sim g, \forall\, g \in G\}$ – pointwise inner automorphisms of G.

Examples of RF and CS groups

Examples of RF groups:

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)
- Groups of automorphisms of a rooted tree of bounded valence

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)
- Groups of automorphisms of a rooted tree of bounded valence

Examples of CS groups:

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)
- Groups of automorphisms of a rooted tree of bounded valence

Examples of CS groups:

- virtually polycyclic groups (Remeslennikov, Formanek)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)
- Groups of automorphisms of a rooted tree of bounded valence

Examples of CS groups:

- virtually polycyclic groups (Remeslennikov, Formanek)
- virtually free groups (Dyer)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)
- Groups of automorphisms of a rooted tree of bounded valence

Examples of CS groups:

- virtually polycyclic groups (Remeslennikov, Formanek)
- virtually free groups (Dyer)
- virtually surface groups (Martino)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)
- Groups of automorphisms of a rooted tree of bounded valence

Examples of CS groups:

- virtually polycyclic groups (Remeslennikov, Formanek)
- virtually free groups (Dyer)
- virtually surface groups (Martino)
- limit groups/fully residually free gps. (Chagas-Zalesskii)

Examples of RF and CS groups

Examples of RF groups:

- virtually polycyclic groups (Hirsch)
- virtually free groups (Schreier, Levi)
- virtually surface groups (Baumslag)
- More generally, f.g. linear groups (Mal'cev)
- Groups of automorphisms of a rooted tree of bounded valence

Examples of CS groups:

- virtually polycyclic groups (Remeslennikov, Formanek)
- virtually free groups (Dyer)
- virtually surface groups (Martino)
- limit groups/fully residually free gps. (Chagas-Zalesskii)
- Let $\mathcal{X}$ be the smallest class of gps. containing virt. free and virt. polyc. gps and closed under cyclic amalgamation. Then each $G \in \mathcal{X}$ is CS (Ribes-Segal-Zalesskii)

Non-CS groups

Every CS group is RF but not vice-versa.

Non-CS groups

Every CS group is RF but not vice-versa.

- $SL_n(\mathbb{Z})$ is not CS if $n \geq 3$ (Remeslennikov, Stebe)

Non-CS groups

Every CS group is RF but not vice-versa.

- $SL_n(\mathbb{Z})$ is not CS if $n \geq 3$ (Remeslennikov, Stebe)
- many subgroups of $\mathbb{F}_2 \times \mathbb{F}_2$ are not CS

Non-CS groups

Every CS group is RF but not vice-versa.

- $SL_n(\mathbb{Z})$ is not CS if $n \geq 3$ (Remeslennikov, Stebe)
- many subgroups of $\mathbb{F}_2 \times \mathbb{F}_2$ are not CS

RF is stable under taking subgroups and finite index overgroups, but

Non-CS groups

Every CS group is RF but not vice-versa.

- $SL_n(\mathbb{Z})$ is not CS if $n \geq 3$ (Remeslennikov, Stebe)
- many subgroups of $\mathbb{F}_2 \times \mathbb{F}_2$ are not CS

RF is stable under taking subgroups and finite index overgroups, but

- there is a f.g. group G with a CS sbgp. $H \leqslant G$ s.t. $|G : H| = 2$ and G is not CS (Goryaga)

Non-CS groups

Every CS group is RF but not vice-versa.

- $SL_n(\mathbb{Z})$ is not CS if $n \geq 3$ (Remeslennikov, Stebe)
- many subgroups of $\mathbb{F}_2 \times \mathbb{F}_2$ are not CS

RF is stable under taking subgroups and finite index overgroups, but

- there is a f.g. group G with a CS sbgp. $H \leqslant G$ s.t. $|G : H| = 2$ and G is not CS (Goryaga)
- there is a f.p. CS group G with a sbgp. $H \leqslant G$ s.t. $|G : H| = 2$ and H is not CS (M.-Martino)

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Definition

The profinite topology on a gp. G is the group topology whose basic open neighborhoods of the identity are subgroups of finite index.

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Definition

The profinite topology on a gp. G is the group topology whose basic open neighborhoods of the identity are subgroups of finite index.

Facts:

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Definition

The profinite topology on a gp. G is the group topology whose basic open neighborhoods of the identity are subgroups of finite index.

Facts:

- A gp. G is RF $\iff$ the prof. top. on G is Hasdorff

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Definition

The profinite topology on a gp. G is the group topology whose basic open neighborhoods of the identity are subgroups of finite index.

Facts:

- A gp. G is RF $\iff$ the prof. top. on G is Hasdorff $\iff$ $\{1\}$ is closed in G

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Definition

The profinite topology on a gp. G is the group topology whose basic open neighborhoods of the identity are subgroups of finite index.

Facts:

- A gp. G is RF $\iff$ the prof. top. on G is Hasdorff $\iff$ $\{1\}$ is closed in G
- G is CS $\iff$ $x^G = \{gxg^{-1} \mid g \in G\}$ is closed in the prof. top. $\forall\, x \in G$

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Definition

The profinite topology on a gp. G is the group topology whose basic open neighborhoods of the identity are subgroups of finite index.

Facts:

- A gp. G is RF $\iff$ the prof. top. on G is Hasdorff $\iff$ $\{1\}$ is closed in G
- G is CS $\iff$ $x^G = \{gxg^{-1} \mid g \in G\}$ is closed in the prof. top. $\forall\, x \in G$
- If $K \leqslant_f G$ then the prof. top. of K is induced by the prof. top. of G

Profinite topology

One of the useful tools for proving that some gp is RF or CS is the profinite topology.

Definition

The profinite topology on a gp. G is the group topology whose basic open neighborhoods of the identity are subgroups of finite index.

Facts:

- A gp. G is RF $\iff$ the prof. top. on G is Hasdorff $\iff$ $\{1\}$ is closed in G
- G is CS $\iff$ $x^G = \{gxg^{-1} \mid g \in G\}$ is closed in the prof. top. $\forall\, x \in G$
- If $K \leqslant_f G$ then the prof. top. of K is induced by the prof. top. of G
- If G is RF then the prof. top. on G is induced by the canonical embedding of G in its profinite completion $\widehat{G}$

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Proof.

$G = \bigcup_{i=1}^{n} g_i K$, hence $x^G = \bigcup_{i=1}^{n} g_i x^K g_i^{-1}$.

□

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Proof.

$G = \bigcup_{i=1}^n g_i K$, hence $x^G = \bigcup_{i=1}^n g_i x^K g_i^{-1}$. Since $|G : K| < \infty$, every closed subset of K is closed in G

□

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Proof.

$G = \bigcup_{i=1}^n g_i K$, hence $x^G = \bigcup_{i=1}^n g_i x^K g_i^{-1}$. Since $|G : K| < \infty$, every closed subset of K is closed in $G \Longrightarrow x^K$ is closed in G

□

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Proof.

$G = \bigcup_{i=1}^n g_i K$, hence $x^G = \bigcup_{i=1}^n g_i x^K g_i^{-1}$. Since $|G : K| < \infty$, every closed subset of K is closed in $G \implies x^K$ is closed in $G \implies x^G$ is closed in G as a finite union of translates of x^K. □

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Let G be a f.g. free group and let $x \in G$ be any element.

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Let G be a f.g. free group and let $x \in G$ be any element. By M. Hall's thm., x is a free generator of some $K \leqslant_f G$,

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Let G be a f.g. free group and let $x \in G$ be any element. By M. Hall's thm., x is a free generator of some $K \leqslant_f G$, and, by the above lemma, to show that x^G is closed in G it's enough to prove that x^K is closed in K.

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Let G be a f.g. free group and let $x \in G$ be any element. By M. Hall's thm., x is a free generator of some $K \leqslant_f G$, and, by the above lemma, to show that x^G is closed in G it's enough to prove that x^K is closed in K.
Thus, WLOG, we can assume that $x \in X$, where X is a free gen. set of G.

Stebe's proof that free groups are CS, 1

We will now present an argument for proving that a f.g. free group is CS, which is due to Stebe.

Lemma

Suppose that G is a gp., $K \leqslant_f G$ and $x \in K$. If x^K is closed in the prof. top. of K then x^G is closed in the prof. top. of G.

Let G be a f.g. free group and let $x \in G$ be any element. By M. Hall's thm., x is a free generator of some $K \leqslant_f G$, and, by the above lemma, to show that x^G is closed in G it's enough to prove that x^K is closed in K.
Thus, WLOG, we can assume that $x \in X$, where X is a free gen. set of G. I.e., it remains to prove the following

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$.

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \ldots x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$.

□

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n = 1$, i.e., $w \in X^{\pm 1}$ and $w \neq x$

□

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n = 1$, i.e., $w \in X^{\pm 1}$ and $w \neq x \implies w$ and x can be distinguished in the mod-3 abelianization of G: $M = G/[G, G]G^3$, $|M| < \infty$. □

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n \geq 2$, construct a cycle labelled by w, where the oriented edges are labelled by x_i's.

$\square$

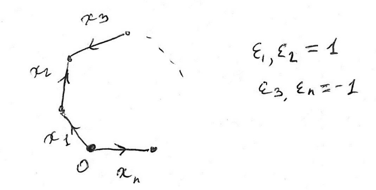
Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n \geq 2$, construct a cycle labelled by w, where the oriented edges are labelled by x_i's.



□

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n \geq 2$, construct a cycle labelled by w, where the oriented edges are labelled by x_i's.
This defines a partial right action of each element of X on the set of n vertices.

□

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n \geq 2$, construct a cycle labelled by w, where the oriented edges are labelled by x_i's.
This defines a partial right action of each element of X on the set of n vertices. Extend it to a full right action so that the generator $x \in X$ does not fix any vertex.

□

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n \geq 2$, construct a cycle labelled by w, where the oriented edges are labelled by x_i's.
This defines a partial right action of each element of X on the set of n vertices. Extend it to a full right action so that the generator $x \in X$ does not fix any vertex.
Thus we get a homom. $\psi : G \to S_n$ s.t. $\psi(w)$ fixes O and $\psi(x)$ does not fix anything.

□

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n \geq 2$, construct a cycle labelled by w, where the oriented edges are labelled by x_i's.
This defines a partial right action of each element of X on the set of n vertices. Extend it to a full right action so that the generator $x \in X$ does not fix any vertex.
Thus we get a homom. $\psi : G \to S_n$ s.t. $\psi(w)$ fixes O and $\psi(x)$ does not fix anything. Hence $\psi(w) \not\sim \psi(x)$ in S_n

□

Stebe's proof that free groups are CS, 2

Lemma

If G is free on a finite set X then $\forall\, x \in X$, x^G is closed in the prof. top. on G.

Proof.

Take any $w \in G \setminus x^G$. WLOG $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is cyclically reduced, where $x_1, \dots x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$.
If $n \geq 2$, construct a cycle labelled by w, where the oriented edges are labelled by x_i's.
This defines a partial right action of each element of X on the set of n vertices. Extend it to a full right action so that the generator $x \in X$ does not fix any vertex.
Thus we get a homom. $\psi : G \to S_n$ s.t. $\psi(w)$ fixes O and $\psi(x)$ does not fix anything. Hence $\psi(w) \not\sim \psi(x)$ in $S_n \implies$ x^G is closed in G.

□

Right angled Artin groups

Definition

Let Γ be a finite simplicial graph. The right angled Artin group (RAAG) $A(\Gamma)$ is defined by

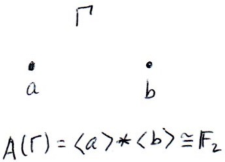
$$A(\Gamma) = \langle v \in V\Gamma \,\|\, uv = vu \text{ if } (u,v) \in E\Gamma \rangle.$$

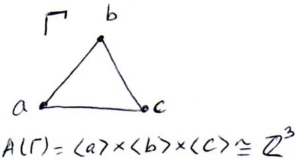
Right angled Artin groups

Definition

Let Γ be a finite simplicial graph. The right angled Artin group (RAAG) $A(\Gamma)$ is defined by

$$A(\Gamma) = \langle v \in V\Gamma \,\|\, uv = vu \text{ if } (u,v) \in E\Gamma \rangle.$$



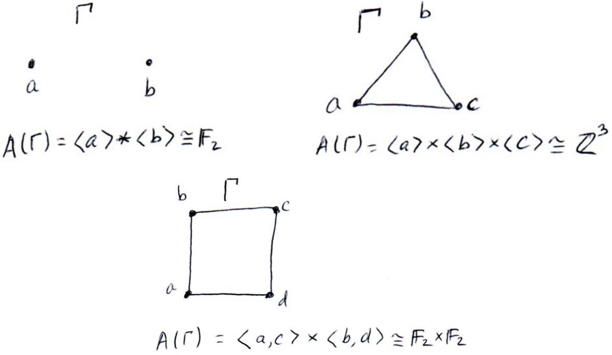


Right angled Artin groups

Definition

Let Γ be a finite simplicial graph. The right angled Artin group (RAAG) $A(\Gamma)$ is defined by

$$A(\Gamma) = \langle v \in V\Gamma \,\|\, uv = vu \text{ if } (u,v) \in E\Gamma \rangle.$$

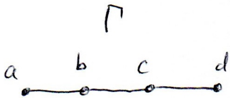


Right angled Artin groups

Definition

Let Γ be a finite simplicial graph. The right angled Artin group (RAAG) $A(\Gamma)$ is defined by

$$A(\Gamma) = \langle v \in V\Gamma \,\|\, uv = vu \text{ if } (u, v) \in E\Gamma \rangle.$$



$$A(\Gamma) = \langle a, b, c, d \,\|\, [a,b] = [b,c] = [c,d] = 1 \rangle$$

Conjugacy separability of RAAGs

Definition

A group G is hereditarily conjugacy separable (HCS) if each $K \leqslant_f G$ is CS.

Definition

A group G is hereditarily conjugacy separable (HCS) if each $K \leqslant_f G$ is CS.

HCS is strictly stronger than CS.

Definition

A group G is hereditarily conjugacy separable (HCS) if each $K \leqslant_f G$ is CS.

HCS is strictly stronger than CS.

Theorem (M.)

Right angled Artin groups are HCS.

Conjugacy separability of RAAGs

Definition

A group G is hereditarily conjugacy separable (HCS) if each $K \leqslant_f G$ is CS.

HCS is strictly stronger than CS.

Theorem (M.)

Right angled Artin groups are HCS.

The proof is quite long; one of the main ingredients is the decomposition of RAAGs as special HNN-extensions.

Special HNN-extensions

Definition

Let B be a gp. and $H \leqslant B$. The special HNN-extension of B w.r.t. H is

$$A = \langle B, t \,\|\, tht^{-1} = h,\ \forall\, h \in H \rangle.$$

Special HNN-extensions

Definition

Let B be a gp. and $H \leqslant B$. The special HNN-extension of B w.r.t. H is

$$A = \langle B, t \,\|\, tht^{-1} = h,\ \forall\, h \in H \rangle.$$

Remark

If $A = A(\Gamma)$ is a RAAG and $t \in V\Gamma$ then A splits as a special HNN-extension of $B = \langle V\Gamma \setminus \{t\} \rangle$ w.r.t. $H = \langle \mathrm{link}(t) \rangle$.

Special HNN-extensions

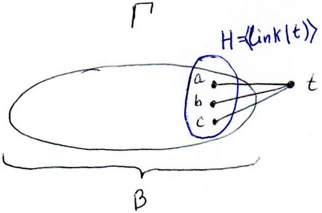
Definition

Let B be a gp. and $H \leqslant B$. The special HNN-extension of B w.r.t. H is

$$A = \langle B, t \,\|\, tht^{-1} = h,\ \forall\, h \in H \rangle.$$

Remark

If $A = A(\Gamma)$ is a RAAG and $t \in V\Gamma$ then A splits as a special HNN-extension of $B = \langle V\Gamma \setminus \{t\} \rangle$ w.r.t. $H = \langle \mathrm{link}(t) \rangle$.



$$A(\Gamma) = \langle B, t \,\|\, [t,a] = [t,b] = [t,c] = 1 \rangle$$
$$= \langle B, t \,\|\, tht^{-1} = h,\ \forall h \in \langle a,b,c \rangle \rangle$$

Special HNN-extensions

Definition

Let B be a gp. and $H \leqslant B$. The special HNN-extension of B w.r.t. H is

$$A = \langle B, t \,\|\, tht^{-1} = h,\ \forall\, h \in H \rangle.$$

Remark

If $A = A(\Gamma)$ is a RAAG and $t \in V\Gamma$ then A splits as a special HNN-extension of $B = \langle V\Gamma \setminus \{t\} \rangle$ w.r.t. $H = \langle \mathrm{link}(t) \rangle$.

Remark (Functoriality)

Let A be the spec. HNN-ext. of B w.r.t. $H \leqslant B$. Then every homom. $\phi : B \to L$ extends to a homom. $\hat{\phi} : A \to M$, where M is the spec. HNN-ext. of L w.r.t. $\phi(H)$.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\} \rangle$.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\}\rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\} \rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

By induction, the profinite top. on B is "nice",

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\}\rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

By induction, the profinite top. on B is "nice", and any homom. $\phi : B \to L$, with $|L| < \infty$, can be extended to a homom. $\hat{\phi} : A \to M$, where M is virt. free as an HNN-ext. of the finite gp. L.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\} \rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

By induction, the profinite top. on B is "nice", and any homom. $\phi : B \to L$, with $|L| < \infty$, can be extended to a homom. $\hat{\phi} : A \to M$, where M is virt. free as an HNN-ext. of the finite gp. L.

Use criteria of conjugacy in special HNN-ext. to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\} \rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

By induction, the profinite top. on B is "nice", and any homom. $\phi : B \to L$, with $|L| < \infty$, can be extended to a homom. $\hat{\phi} : A \to M$, where M is virt. free as an HNN-ext. of the finite gp. L.

Use criteria of conjugacy in special HNN-ext. to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M.

By Dyer's thm., virt. free gps. are CS, so $\exists\, \eta : M \to P$, $|P| < \infty$, s.t. $\eta(\hat{\phi}(x)) \not\sim \eta(\hat{\phi}(y))$ in P.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\} \rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

By induction, the profinite top. on B is "nice", and any homom. $\phi : B \to L$, with $|L| < \infty$, can be extended to a homom. $\hat{\phi} : A \to M$, where M is virt. free as an HNN-ext. of the finite gp. L.

Use criteria of conjugacy in special HNN-ext. to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M.

By Dyer's thm., virt. free gps. are CS, so $\exists \eta : M \to P$, $|P| < \infty$, s.t. $\eta(\hat{\phi}(x)) \not\sim \eta(\hat{\phi}(y))$ in P.

Hence $\psi(x) \not\sim \psi(y)$ in P, where $\psi = \eta \circ \hat{\phi} : A \to P$, as required. $\square$

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\}\rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

By induction, the profinite top. on B is "nice", and any homom. $\phi : B \to L$, with $|L| < \infty$, can be extended to a homom. $\hat{\phi} : A \to M$, where M is virt. free as an HNN-ext. of the finite gp. L.

Use criteria of conjugacy in special HNN-ext. to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M.

By Dyer's thm., virt. free gps. are CS, so $\exists\, \eta : M \to P$, $|P| < \infty$, s.t. $\eta(\hat{\phi}(x)) \not\sim \eta(\hat{\phi}(y))$ in P.

Hence $\psi(x) \not\sim \psi(y)$ in P, where $\psi = \eta \circ \hat{\phi} : A \to P$, as required. $\square$

Problem: to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M one needs more properties than simply CS of B.

Idea of the proof that RAAGs are CS

Let $A = A(\Gamma)$ be a RAAG. We prove that A is CS by induction on $|V\Gamma|$.

Let $x, y \in A$ be s.t. $x \not\sim y$. Choose $t \in V\Gamma$ and use the decomp. of A as a spec. HNN-ext. of $B = \langle V\Gamma \setminus \{t\}\rangle$.

Note that $B \cong A(\Gamma')$ is also a RAAG, where $\Gamma' = \Gamma \setminus \{t\}$.

By induction, the profinite top. on B is "nice", and any homom. $\phi : B \to L$, with $|L| < \infty$, can be extended to a homom. $\hat{\phi} : A \to M$, where M is virt. free as an HNN-ext. of the finite gp. L.

Use criteria of conjugacy in special HNN-ext. to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M.

By Dyer's thm., virt. free gps. are CS, so $\exists\, \eta : M \to P$, $|P| < \infty$, s.t. $\eta(\hat{\phi}(x)) \not\sim \eta(\hat{\phi}(y))$ in P.

Hence $\psi(x) \not\sim \psi(y)$ in P, where $\psi = \eta \circ \hat{\phi} : A \to P$, as required. □

Problem: to ensure that $\hat{\phi}(x) \not\sim \hat{\phi}(y)$ in M one needs more properties than simply CS of B. That's why we have to prove the stronger result, that A is HCS.

Virtual retracts of RAAGs

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Virtual retracts of RAAGs

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Virtual retracts of RAAGs

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Indeed, if $x, y \in R$ and $x \not\sim_R y$ then $x \not\sim_G y$, as R is a retract,

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Indeed, if $x, y \in R$ and $x \not\sim_R y$ then $x \not\sim_G y$, as R is a retract, so one can separate y from x^G in a finite quotient of G, which induces required finite quotient of R.

Virtual retracts of RAAGs

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Indeed, if $x, y \in R$ and $x \not\sim_R y$ then $x \not\sim_G y$, as R is a retract, so one can separate y from x^G in a finite quotient of G, which induces required finite quotient of R.

$H \leqslant G$ is a virtual retract if $\exists\, K \leqslant_f G$ s.t. H is retract of K.

Virtual retracts of RAAGs

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Indeed, if $x, y \in R$ and $x \not\sim_R y$ then $x \not\sim_G y$, as R is a retract, so one can separate y from x^G in a finite quotient of G, which induces required finite quotient of R.

$H \leqslant G$ is a virtual retract if $\exists\, K \leqslant_f G$ s.t. H is retract of K.

Corollary

Virtual retracts of RAAGs are CS.

Virtual retracts of RAAGs

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Indeed, if $x, y \in R$ and $x \not\sim_R y$ then $x \not\sim_G y$, as R is a retract, so one can separate y from x^G in a finite quotient of G, which induces required finite quotient of R.

$H \leqslant G$ is a virtual retract if $\exists\, K \leqslant_f G$ s.t. H is retract of K.

Corollary

Virtual retracts of RAAGs are CS.

It is a current theme in Geometric Group Theory to show that, up to finite index, many groups are virtual retracts RAAGs.

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Indeed, if $x, y \in R$ and $x \not\sim_R y$ then $x \not\sim_G y$, as R is a retract, so one can separate y from x^G in a finite quotient of G, which induces required finite quotient of R.

$H \leqslant G$ is a virtual retract if $\exists\, K \leqslant_f G$ s.t. H is retract of K.

Corollary

Virtual retracts of RAAGs are CS.

It is a current theme in Geometric Group Theory to show that, up to finite index, many groups are virtual retracts RAAGs. This includes virtually compact special (VCS) groups of Haglund and Wise.

Virtual retracts of RAAGs

Recall that $R \leqslant G$ is a retract if $\exists\, \rho : G \to R$ s.t. $\rho|_R = \mathrm{Id}_R$.

Remark

A retract of a CS group is itself CS.

Indeed, if $x, y \in R$ and $x \not\sim_R y$ then $x \not\sim_G y$, as R is a retract, so one can separate y from x^G in a finite quotient of G, which induces required finite quotient of R.

$H \leqslant G$ is a virtual retract if $\exists\, K \leqslant_f G$ s.t. H is retract of K.

Corollary

Virtual retracts of RAAGs are CS.

It is a current theme in Geometric Group Theory to show that, up to finite index, many groups are virtual retracts RAAGs. This includes virtually compact special (VCS) groups of Haglund and Wise.

VCS groups usually act on non-positively curved spaces (e.g., cube complexes).

Virtually compact special groups

Theorem

The following groups are VCS:

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*
- 1*-relator groups with torsion (Wise)*

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*
- 1*-relator groups with torsion (Wise)*
- *F.p. $C'(1/6)$ small cancellation groups (Wise, Agol)*

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*
- 1*-relator groups with torsion (Wise)*
- *F.p.* $C'(1/6)$ *small cancellation groups (Wise, Agol)*
- *Limit groups/fully residually free groups (Wise)*

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*
- 1*-relator groups with torsion (Wise)*
- *F.p.* $C'(1/6)$ *small cancellation groups (Wise, Agol)*
- *Limit groups/fully residually free groups (Wise)*
- *Fund. groups of closed hyperbolic* 3*-manifolds (Agol)*

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*
- 1*-relator groups with torsion (Wise)*
- *F.p.* $C'(1/6)$ *small cancellation groups (Wise, Agol)*
- *Limit groups/fully residually free groups (Wise)*
- *Fund. groups of closed hyperbolic* 3*-manifolds (Agol)*
- *Word hyperbolic (f.g. free)-by-cyclic groups (Hagen-Wise)*

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*
- 1*-relator groups with torsion (Wise)*
- *F.p.* $C'(1/6)$ *small cancellation groups (Wise, Agol)*
- *Limit groups/fully residually free groups (Wise)*
- *Fund. groups of closed hyperbolic* 3*-manifolds (Agol)*
- *Word hyperbolic (f.g. free)-by-cyclic groups (Hagen-Wise)*

Thus each of the above groups has a CS sbgp. of finite index.

Virtually compact special groups

Theorem

The following groups are VCS:

- *Many (e.g., word hyperbolic) Coxeter groups (Haglund-Wise)*
- 1*-relator groups with torsion (Wise)*
- *F.p.* $C'(1/6)$ *small cancellation groups (Wise, Agol)*
- *Limit groups/fully residually free groups (Wise)*
- *Fund. groups of closed hyperbolic* 3*-manifolds (Agol)*
- *Word hyperbolic (f.g. free)-by-cyclic groups (Hagen-Wise)*

Thus each of the above groups has a CS sbgp. of finite index.

But CS may not pass to a finite index overgroup! So, there is still work to be done.

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic* 3*-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*
- *Bianchi groups, non-uniform lattices in $SL_2(\mathbb{C})$ (Chagas-Zalesskii)*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*
- *Bianchi groups, non-uniform lattices in $SL_2(\mathbb{C})$ (Chagas-Zalesskii)*
- *Fund. groups of compact orientable 3-manifolds (Hamilton-Wilton-Zalesskii)*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*
- *Bianchi groups, non-uniform lattices in $SL_2(\mathbb{C})$ (Chagas-Zalesskii)*
- *Fund. groups of compact orientable 3-manifolds (Hamilton-Wilton-Zalesskii)*
- *Word hyperbolic VCS groups (M.-Zalesskii), in particular:*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*
- *Bianchi groups, non-uniform lattices in $SL_2(\mathbb{C})$ (Chagas-Zalesskii)*
- *Fund. groups of compact orientable 3-manifolds (Hamilton-Wilton-Zalesskii)*
- *Word hyperbolic VCS groups (M.-Zalesskii), in particular:*
  - *1-relator groups with torsion*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*
- *Bianchi groups, non-uniform lattices in* $SL_2(\mathbb{C})$ *(Chagas-Zalesskii)*
- *Fund. groups of compact orientable 3-manifolds (Hamilton-Wilton-Zalesskii)*
- *Word hyperbolic VCS groups (M.-Zalesskii), in particular:*
  - *1-relator groups with torsion*
  - *word hyperbolic Coxeter gps.*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*
- *Bianchi groups, non-uniform lattices in $SL_2(\mathbb{C})$ (Chagas-Zalesskii)*
- *Fund. groups of compact orientable 3-manifolds (Hamilton-Wilton-Zalesskii)*
- *Word hyperbolic VCS groups (M.-Zalesskii), in particular:*
  - *1-relator groups with torsion*
  - *word hyperbolic Coxeter gps.*
  - *f.p. $C'(1/6)$ small cancellation groups*

Conjugacy separability of VCS groups

HCS of RAAGs plays a central role in proving

Theorem

The following groups are CS:

- *Torsion-free word hyperbolic VCS groups (M.), in particular:*
  - *fund. gps. of closed hyperbolic 3-manifolds*
  - *word hyperbolic (f.g. free)-by-cyclic groups*
- *Most even Coxeter groups (Caprace-M.)*
- *Bianchi groups, non-uniform lattices in $SL_2(\mathbb{C})$ (Chagas-Zalesskii)*
- *Fund. groups of compact orientable 3-manifolds (Hamilton-Wilton-Zalesskii)*
- *Word hyperbolic VCS groups (M.-Zalesskii), in particular:*
  - *1-relator groups with torsion*
  - *word hyperbolic Coxeter gps.*
  - *f.p. $C'(1/6)$ small cancellation groups*
  - *uniform lattices in $SL_2(\mathbb{C})$*