# The SQ-universality and residual properties of relatively hyperbolic groups

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**Abstract**

In this paper we study residual properties of relatively hyperbolic groups. In particular, we show that if a group $G$ is non-elementary and hyperbolic relative to a collection of proper subgroups, then $G$ is SQ-universal.

## 1 Introduction

The notion of a group hyperbolic relative to a collection of subgroups was originally suggested by Gromov [9] and since then it has been elaborated from different points of view [3, 6, 5, 21]. The class of relatively hyperbolic groups includes many examples. For instance, if $M$ is a complete finite-volume manifold of pinched negative sectional curvature, then $\pi_1(M)$ is hyperbolic with respect to the cusp subgroups [3, 6]. More generally, if $G$ acts isometrically and properly discontinuously on a proper hyperbolic metric space $X$ so that the induced action of $G$ on $\partial X$ is geometrically finite, then $G$ is hyperbolic relative to the collection of maximal parabolic subgroups [3]. Groups acting on $CAT(0)$ spaces with isolated flats are hyperbolic relative to the collection of flat stabilizers [13]. Algebraic examples of relatively hyperbolic groups include free products and their small cancellation quotients [21], fully residually free groups (or Sela's limit groups) [4], and, more generally, groups acting freely on $\mathbb{R}^n$-trees [10].

The main goal of this paper is to study residual properties of relatively hyperbolic groups. Recall that a group $G$ is called *SQ-universal* if every countable group can be embedded into a quotient of $G$ [25]. It is straightforward to see that any SQ-universal group contains an infinitely generated free subgroup. Furthermore, since the set of all finitely generated groups is uncountable and every single quotient of $G$ contains (at most) countably many finitely generated subgroups, every SQ-universal group has uncountably many non-isomorphic quotients. Thus the property of being SQ-universal may, in a very rough sense, be considered as an indication of "largeness" of a group.

The first non-trivial example of an SQ-universal group was provided by Higman, Neumann and Neumann [11], who proved that the free group of rank 2 is SQ-universal. Presently

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many other classes of groups are known to be SQ-universal: various HNN-extensions and amalgamated products [7, 15, 24], groups of deficiency 2 [2], most $C(3)\,\&\,T(6)$-groups [12], etc. The SQ-universality of non-elementary hyperbolic groups was proved by Olshanskii in [19]. On the other hand, for relatively hyperbolic groups, there are some partial results. Namely, in [8] Fine proved the SQ-universality of certain Kleinian groups. The case of fundamental groups of hyperbolic 3-manifolds was studied by Ratcliffe in [23].

In this paper we prove the SQ-universality of relatively hyperbolic groups in the most general settings. Let a group $G$ be hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ (called *peripheral subgroups*). We say that $G$ is *properly hyperbolic relative to* $\{H_\lambda\}_{\lambda\in\Lambda}$ (or $G$ is a *PRH group* for brevity), if $H_\lambda \neq G$ for all $\lambda \in \Lambda$. Recall that a group is *elementary*, if it contains a cyclic subgroup of finite index. We observe that every non-elementary PRH group has a unique maximal finite normal subgroup denoted by $E_G(G)$ (see Lemmas 4.3 and 3.3 below).

**Theorem 1.1.** *Suppose that a group $G$ is non-elementary and properly relatively hyperbolic with respect to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then for each finitely generated group $R$, there exists a quotient group $Q$ of $G$ and an embedding $R \hookrightarrow Q$ such that:*

1. *$Q$ is properly relatively hyperbolic with respect to the collection $\{\psi(H_\lambda)\}_{\lambda\in\Lambda} \cup \{R\}$ where $\psi\colon G \to Q$ denotes the natural epimorphism;*
2. *For each $\lambda \in \Lambda$, we have $H_\lambda \cap \ker(\psi) = H_\lambda \cap E_G(G)$, that is, $\psi(H_\lambda)$ is naturally isomorphic to $H_\lambda/(H_\lambda \cap E_G(G))$.*

In general, we can not require the epimorphism $\psi$ to be injective on every $H_\lambda$. Indeed, it is easy to show that a finite normal subgroup of a relatively hyperbolic group must be contained in each infinite peripheral subgroup (see Lemma 4.4). Thus the image of $E_G(G)$ in $Q$ will have to be inside $R$ whenever $R$ is infinite. If, in addition, the group $R$ is torsion-free, the latter inclusion implies $E_G(G) \leq \ker(\psi)$. This would be the case if one took $G = F_2 \times \mathbb{Z}/(2\mathbb{Z})$ and $R = \mathbb{Z}$, where $F_2$ denotes the free group of rank 2 and $G$ is properly hyperbolic relative to its subgroup $\mathbb{Z}/(2\mathbb{Z}) = E_G(G)$.

Since any countable group is embeddable into a finitely generated group, we obtain the following.

**Corollary 1.2.** *Any non-elementary PRH group is SQ-universal.*

Let us mention a particular case of Corollary 1.2. In [7] the authors asked whether every finitely generated group with infinite number of ends is SQ-universal. The celebrated Stallings theorem [26] states that a finitely generated group has infinite number of ends if and only if it splits as a nontrivial HNN-extension or amalgamated product over a finite subgroup. The case of amalgamated products was considered by Lossov who provided the positive answer in [15]. Corollary 1.2 allows us to answer the question in the general case. Indeed, every group with infinite number of ends is non-elementary and properly relatively hyperbolic, since the action of such a group on the corresponding Bass-Serre tree satisfies Bowditch's definition of relative hyperbolicity [3].

**Corollary 1.3.** *A finitely generated group with infinite number of ends is SQ-universal.*

The methods used in the proof of Theorem 1.1 can also be applied to obtain other results:

**Theorem 1.4.** *Any two finitely generated non-elementary PRH groups $G_1, G_2$ have a common non-elementary PRH quotient $Q$. Moreover, $Q$ can be obtained from the free product $G_1 * G_2$ by adding finitely many relations.*

In [18] Olshanskii proved that any non-elementary hyperbolic group has a non-trivial finitely presented quotient without proper subgroups of finite index. This result was used by Lubotzky and Bass [1] to construct representation rigid linear groups of non-arithmetic type thus solving in negative the Platonov Conjecture. Theorem 1.4 yields a generalization of Olshanskii's result.

**Definition 1.5.** Given a class of groups $\mathcal{G}$, we say that a group $R$ is *residually incompatible with* $\mathcal{G}$ if for any group $A \in \mathcal{G}$, any homomorphism $R \to A$ has a trivial image.

If $G$ and $R$ are finitely presented groups, $G$ is properly relatively hyperbolic, and $R$ is residually incompatible with a class of groups $\mathcal{G}$, we can apply Theorem 1.4 to $G_1 = G$ and $G_2 = R * R$. Obviously, the obtained common quotient of $G_1$ and $G_2$ is finitely presented and residually incompatible with $\mathcal{G}$.

**Corollary 1.6.** *Let $\mathcal{G}$ be a class of groups. Suppose that there exists a finitely presented group $R$ that is residually incompatible with $\mathcal{G}$. Then every finitely presented non-elementary PRH group has a non-trivial finitely presented quotient group that is residually incompatible with $\mathcal{G}$.*

Recall that there are finitely presented groups having no non-trivial recursively presented quotients with decidable word problem [16]. Applying the previous corollary to the class $\mathcal{G}$ of all recursively presented groups with decidable word problem, we obtain the following result.

**Corollary 1.7.** *Every non-elementary finitely presented PRH group has an infinite finitely presented quotient group $Q$ such that the word problem is undecidable in each non-trivial quotient of $Q$.*

In particular, $Q$ has no proper subgroups of finite index. The reader can easily check that Corollary 1.6 can also be applied to the classes of all torsion (torsion-free, Noetherian, Artinian, amenable, etc.) groups.

## 2 Relatively hyperbolic groups

We recall the definition of relatively hyperbolic groups suggested in [21] (for equivalent definitions in the case of finitely generated groups see [3, 5, 6]). Let $G$ be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a fixed collection of subgroups of $G$ (called *peripheral subgroups*), $X$ a subset of $G$. We say that $X$ is a *relative generating set of* $G$ with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ if $G$ is generated by $X$ together with the union of all $H_\lambda$ (for convenience, we always assume that $X = X^{-1}$). In this situation the group $G$ can be considered as a quotient of the free product

$$F = (*_{\lambda\in\Lambda} H_\lambda) * F(X), \qquad (1)$$

where $F(X)$ is the free group with the basis $X$. Suppose that $\mathcal{R}$ is a subset of $F$ such that the kernel of the natural epimorphism $F \to G$ is a normal closure of $\mathcal{R}$ in the group $F$, then we say that $G$ has *relative presentation*

$$\langle X,\ \{H_\lambda\}_{\lambda\in\Lambda} \mid R = 1,\ R \in \mathcal{R}\rangle. \tag{2}$$

If sets $X$ and $\mathcal{R}$ are finite, the presentation (2) is said to be *relatively finite*.

**Definition 2.1.** We set

$$\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} (H_\lambda \setminus \{1\}). \tag{3}$$

A group $G$ is *relatively hyperbolic with respect to a collection of subgroups* $\{H_\lambda\}_{\lambda\in\Lambda}$, if $G$ admits a relatively finite presentation (2) with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ satisfying a *linear relative isoperimetric inequality*. That is, there exists $C > 0$ satisfying the following condition. For every word $w$ in the alphabet $X \cup \mathcal{H}$ representing the identity in the group $G$, there exists an expression

$$w =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i \tag{4}$$

with the equality in the group $F$, where $R_i \in \mathcal{R}$, $f_i \in F$, for $i = 1, \ldots, k$, and $k \leq C\|w\|$, where $\|w\|$ is the length of the word $w$. This definition is independent of the choice of the (finite) generating set $X$ and the (finite) set $\mathcal{R}$ in (2).

For a combinatorial path $p$ in the Cayley graph $\Gamma(G, X\cup\mathcal{H})$ of $G$ with respect to $X\cup\mathcal{H}$, $p_-$, $p_+$, $l(p)$, and $\mathbf{Lab}\,(p)$ will denote the initial point, the ending point, the length (that is, the number of edges) and the label of $p$ respectively. Further, if $\Omega$ is a subset of $G$ and $g \in \langle\Omega\rangle \leq G$, then $|g|_\Omega$ will be used to denote the length of a shortest word in $\Omega^{\pm 1}$ representing $g$.

Let us recall some terminology introduced in [21]. Suppose $q$ is a path in $\Gamma(G, X \cup \mathcal{H})$.

**Definition 2.2.** A subpath $p$ of $q$ is called an $H_\lambda$-*component* for some $\lambda \in \Lambda$ (or simply a *component*) of $q$, if the label of $p$ is a word in the alphabet $H_\lambda \setminus \{1\}$ and $p$ is not contained in a bigger subpath of $q$ with this property.

Two components $p_1, p_2$ of a path $q$ in $\Gamma(G, X \cup \mathcal{H})$ are called *connected* if they are $H_\lambda$-components for the same $\lambda \in \Lambda$ and there exists a path $c$ in $\Gamma(G, X\cup\mathcal{H})$ connecting a vertex of $p_1$ to a vertex of $p_2$ such that $\mathbf{Lab}\,(c)$ entirely consists of letters from $H_\lambda$. In algebraic terms this means that all vertices of $p_1$ and $p_2$ belong to the same coset $gH_\lambda$ for a certain $g \in G$. We can always assume $c$ to have length at most 1, as every nontrivial element of $H_\lambda$ is included in the set of generators. An $H_\lambda$-component $p$ of a path $q$ is called *isolated* if no distinct $H_\lambda$-component of $q$ is connected to $p$. A path $q$ is said to be *without backtracking* if all its components are isolated.

The next lemma is a simplification of Lemma 2.27 from [21].

**Lemma 2.3.** *Suppose that a group $G$ is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then there exists a finite subset $\Omega \subseteq G$ and a constant $K \geq 0$ such that* *the following condition holds. Let $q$ be a cycle in $\Gamma(G, X \cup \mathcal{H})$, $p_1, \ldots, p_k$ a set of isolated $H_\lambda$-components of $q$ for some $\lambda \in \Lambda$, $g_1, \ldots, g_k$ elements of $G$ represented by labels $\mathbf{Lab}(p_1), \ldots, \mathbf{Lab}(p_k)$ respectively. Then $g_1, \ldots, g_k$ belong to the subgroup $\langle \Omega \rangle \leq G$ and the word lengths of $g_i$'s with respect to $\Omega$ satisfy the inequality*

$$\sum_{i=1}^{k} |g_i|_\Omega \leq Kl(q).$$

# 3 Suitable subgroups of relatively hyperbolic groups

Throughout this section let $G$ be a group which is properly hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$, $X$ a finite relative generating set of $G$, and $\Gamma(G, X \cup \mathcal{H})$ the Cayley graph of $G$ with respect to the generating set $X \cup \mathcal{H}$, where $\mathcal{H}$ is given by (3). Recall that an element $g \in G$ is called *hyperbolic* if it is not conjugate to an element of some $H_\lambda$, $\lambda \in \Lambda$. The following description of elementary subgroups of $G$ was obtained in [20].

**Lemma 3.1.** *Let $g$ be a hyperbolic element of infinite order of $G$. Then the following conditions hold.*

1. *The element $g$ is contained in a unique maximal elementary subgroup $E_G(g)$ of $G$, where*
$$E_G(g) = \{f \in G \ : \ f^{-1}g^n f = g^{\pm n} \text{ for some } n \in \mathbb{N}\}. \tag{5}$$
2. *The group $G$ is hyperbolic relative to the collection $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{E_G(g)\}$.*

Given a subgroup $S \leq G$, we denote by $S^0$ the set of all hyperbolic elements of $S$ of infinite order. Recall that two elements $f, g \in G^0$ are said to be *commensurable* (in G) if $f^k$ is conjugated to $g^l$ in $G$ for some non-zero integers $k$ and $l$.

**Definition 3.2.** A subgroup $S \leq G$ is called *suitable*, if there exist at least two non-commensurable elements $f_1, f_2 \in S^0$, such that $E_G(f_1) \cap E_G(f_2) = \{1\}$.

If $S^0 \neq \emptyset$, we define
$$E_G(S) = \bigcap_{g\in S^0} E_G(g).$$

**Lemma 3.3.** *If $S \leq G$ is a non-elementary subgroup and $S^0 \neq \emptyset$, then $E_G(S)$ is the maximal finite subgroup of $G$ normalized by $S$.*

*Proof.* Indeed, if a finite subgroup $M \leq G$ is normalized by $S$, then $|S : C_S(M)| < \infty$ where $C_S(M) = \{g \in S \ : \ g^{-1}xg = x, \ \forall\, x \in M\}$. Formula (5) implies that $M \leq E_G(g)$ for every $g \in S^0$, hence $M \leq E_G(S)$.

On the other hand, if $S$ is non-elementary and $S^0 \neq \emptyset$, there exist $h \in S^0$ and $a \in S^0 \setminus E_G(h)$. Then $a^{-1}ha \in S^0$ and the intersection $E_G(a^{-1}ha) \cap E_G(h)$ is finite. Indeed if $E_G(a^{-1}ha) \cap E_G(h)$ were infinite, we would have $(a^{-1}ha)^n = h^k$ for some $n, k \in \mathbb{Z}\setminus\{0\}$, which would contradict to $a \notin E_G(h)$. Hence $E_G(S) \leq E_G(a^{-1}ha) \cap E_G(h)$ is finite. Obviously, $E_G(S)$ is normalized by $S$ in $G$. $\square$

The main result of this section is the following

**Proposition 3.4.** *Suppose that a group $G$ is hyperbolic relative to a collection $\{H_\lambda\}_{\lambda\in\Lambda}$ and $S$ is a subgroup of $G$. Then the following conditions are equivalent.*

*(1) $S$ is suitable;*

*(2) $S^0 \neq \emptyset$ and $E_G(S) = \{1\}$.*

Our proof of Proposition 3.4 will make use of several auxiliary statements below.

**Lemma 3.5** (Lemma 4.4, [20]). *For any $\lambda \in \Lambda$ and any element $a \in G \setminus H_\lambda$, there exists a finite subset $\mathcal{F}_\lambda = \mathcal{F}_\lambda(a) \subseteq H_\lambda$ such that if $h \in H_\lambda \setminus \mathcal{F}_\lambda$, then $ah$ is a hyperbolic element of infinite order.*

It can be seen from Lemma 3.1 that every hyperbolic element $g \in G$ of infinite order is contained inside the elementary subgroup

$$E_G^+(g) = \{f \in G \ : \ f^{-1}g^n f = g^n \text{ for some } n \in \mathbb{N}\} \leq E_G(g),$$

and $|E_G(g) : E_G^+(g)| \leq 2$.

**Lemma 3.6.** *Suppose $g_1, g_2 \in G^0$ are non-commensurable and $A = \langle g_1, g_2\rangle \leq G$. Then there exists an element $h \in A^0$ such that:*

1. *$h$ is not commensurable with $g_1$ and $g_2$;*
2. *$E_G(h) = E_G^+(h) \leq \langle h, E_G(g_1) \cap E_G(g_2)\rangle$. If, in addition, $E_G(g_j) = E_G^+(g_j)$, $j = 1, 2$, then $E_G(h) = E_G^+(h) = \langle h\rangle \times (E_G(g_1) \cap E_G(g_2))$.*

*Proof.* By Lemma 3.1, $G$ is hyperbolic relative to the collection of peripheral subgroups $\mathfrak{C}_1 = \{H_\lambda\}_{\lambda\in\Lambda} \cup \{E_G(g_1)\} \cup \{E_G(g_2)\}$. The center $Z(E_G^+(g_j))$ has finite index in $E_G^+(g_j)$, hence (possibly, after replacing $g_j$ with a power of itself) we can assume that $g_j \in Z(E_G^+(g_j))$, $j = 1, 2$. Using Lemma 3.5 we can find an integer $n_1 \in \mathbb{N}$ such that the element $g_3 = g_2 g_1^{n_1} \in A$ is hyperbolic relatively to $\mathfrak{C}_1$ and has infinite order. Applying Lemma 3.1 again, we achieve hyperbolicity of $G$ relative to $\mathfrak{C}_2 = \mathfrak{C}_1 \cup \{E_G(g_3)\}$. Set $\mathcal{H}' = \bigsqcup_{H\in\mathfrak{C}_2}(H \setminus \{1\})$.

Let $\Omega \subset G$ be the finite subset and $K > 0$ the constant chosen according to Lemma 2.3 (where $G$ is considered to be relatively hyperbolic with respect to $\mathfrak{C}_2$). Using Lemma 3.5 two more times, we can find numbers $m_1, m_2, m_3 \in \mathbb{N}$ such that

$$g_i^{m_i} \notin \{y \in \langle\Omega\rangle \ : \ |y|_\Omega \leq 21K\}, \quad i = 1, 2, 3, \tag{6}$$

and $h = g_1^{m_1} g_3^{m_3} g_2^{m_2} \in A$ is a hyperbolic element (with respect to $\mathfrak{C}_2$) and has infinite order. Indeed, first we choose $m_1$ to satisfy (6). By Lemma 3.5, there is $m_3$ satisfying (6), so that $g_1^{m_1} g_3^{m_3} \in A^0$. Similarly $m_2$ can be chosen sufficiently big to satisfy (6) and $g_1^{m_1} g_3^{m_3} g_2^{m_2} \in A^0$. In particular, $h$ will be non-commensurable with $g_j$, $j = 1, 2$ (otherwise, there would exist $f \in G$ and $n \in \mathbb{N}$ such that $f^{-1}h^n f \in E(g_j)$, implying $h \in fE(g_j)f^{-1}$ by Lemma 3.1 and contradicting the hyperbolicity of $h$).

Consider a path $q$ labelled by the word $(g_1^{m_1}g_3^{m_3}g_2^{m_2})^l$ in $\Gamma(G, X\cup\mathcal{H}')$ for some $l\in\mathbb{Z}\backslash\{0\}$, where each $g_i^{m_i}$ is treated as a single letter from $\mathcal{H}'$. After replacing $q$ with $q^{-1}$, if necessary, we assume that $l\in\mathbb{N}$. Let $p_1,\ldots,p_{3l}$ be all components of $q$; by the construction of $q$, we have $l(p_j)=1$ for each $j$. Suppose not all of these components are isolated. Then one can find indices $1\le s<t\le 3l$ and $i\in\{1,2,3\}$ such that $p_s$ and $p_t$ are $E_G(g_i)$-components of $q$, $(p_t)_-$ and $(p_s)_+$ are connected by a path $r$ with $\mathbf{Lab}\,(r)\in E_G(g_i)$, $l(r)\le 1$, and $(t-s)$ is minimal with this property. To simplify the notation, assume that $i=1$ (the other two cases are similar). Then $p_{s+1},p_{s+4},\ldots,p_{t-2}$ are isolated $E_G(g_3)$-components of the cycle $p_{s+1}p_{s+2}\ldots p_{t-1}r$, and there are exactly $(t-s)/3\ge 1$ of them. Applying Lemma 2.3, we obtain $g_3^{m_3}\in\langle\Omega\rangle$ and

$$\frac{t-s}{3}|g_3^{m_3}|_\Omega\le K(t-s).$$

Hence $|g_3^{m_3}|_\Omega\le 3K$, contradicting (6). Therefore two distinct components of $q$ can not be connected with each other; that is, the path $q$ is without backtracking.

To finish the proof of Lemma 3.6 we need an auxiliary statement below. Denote by $\mathcal{W}$ the set of all subwords of words $(g_1^{m_1}g_3^{m_3}g_2^{m_2})^l$, $l\in\mathbb{Z}$ (where $g_i^{\pm m_i}$ is treated as a single letter from $\mathcal{H}'$). Consider an arbitrary cycle $o=rqr'q'$ in $\Gamma(G,X\cup\mathcal{H}')$, where $\mathbf{Lab}\,(q),\mathbf{Lab}\,(q')\in\mathcal{W}$; and set $C=\max\{l(r),l(r')\}$. Let $p$ be a component of $q$ (or $q'$). We will say that $p$ is *regular* if it is not an isolated component of $o$. As $q$ and $q'$ are without backtracking, this means that $p$ is either connected to some component of $q'$ (respectively $q$), or to a component of $r$, or $r'$.

**Lemma 3.7.** *In the above notations*

(a) *if $C\le 1$ then every component of $q$ or $q'$ is regular;*

(b) *if $C\ge 2$ then each of $q$ and $q'$ can have at most $15C$ components which are not regular.*

*Proof.* Assume the contrary to (a). Then one can choose a cycle $o=rqr'q'$ with $l(r),l(r')\le 1$, having at least one $E(g_i)$-isolated component on $q$ or $q'$ for some $i\in\{1,2,3\}$, and such that $l(q)+l(q')$ is minimal. Clearly the latter condition implies that each component of $q$ or $q'$ is an isolated component of $o$. Therefore $q$ and $q'$ together contain $k$ distinct $E(g_i)$-components of $o$ where $k\ge 1$ and $k\ge\lfloor l(q)/3\rfloor+\lfloor l(q')/3\rfloor$. Applying Lemma 2.3 we obtain $g_i^{m_i}\in\langle\Omega\rangle$ and $k|g_i^{m_i}|_\Omega\le K(l(q)+l(q')+2)$, therefore $|g_i^{m_i}|_\Omega\le 11K$, contradicting the choice of $m_i$ in (6).

Let us prove (b). Suppose that $C\ge 2$ and $q$ contains more than $15C$ isolated components of $o$. We consider two cases:

**Case 1**. No component of $q$ is connected to a component of $q'$. Then a component of $q$ or $q'$ can be regular only if it is connected to a component of $r$ or $r'$. Since $q$ and $q'$ are without backtracking, two distinct components of $q$ or $q'$ can not be connected to the same component of $r$ (or $r'$). Hence $q$ and $q'$ together can contain at most $2C$ regular components. Thus there is an index $i\in\{1,2,3\}$ such that the cycle $o$ has $k$ isolated $E(g_i)$-components, where $k\ge\lfloor l(q)/3\rfloor+\lfloor l(q')/3\rfloor-2C\ge\lfloor 5C\rfloor-2C>2C>3$. By Lemma 2.3, $g_i^{m_i}\in\langle\Omega\rangle$ and $k|g_i^{m_i}|_\Omega\le K(l(q)+l(q')+2C)$, hence

$$|g_i^{m_i}|_\Omega\le K\frac{3(\lfloor l(q)/3\rfloor+1)+3(\lfloor l(q')/3\rfloor+1)+2C}{\lfloor l(q)/3\rfloor+\lfloor l(q')/3\rfloor-2C}\le K\left(3+\frac{6+8C}{2C}\right)\le 9K,$$

contradicting the choice of $m_i$ in (6).

**Case 2.** The path $q$ has at least one component which is connected to a component of $q'$. Let $p_1, \ldots, p_{l(q)}$ denote the sequence of all components of $q$. By part (a), if $p_s$ and $p_t$, $1 \leq s \leq t \leq l(q)$, are connected to components of $q'$, then for any $j$, $s \leq j \leq t$, $p_j$ is regular. We can take $s$ (respectively $t$) to be minimal (respectively maximal) possible. Consequently $p_1, \ldots, p_{s-1}, p_{t+1}, \ldots, p_{l(q)}$ will contain the set of all isolated components of $o$ that belong to $q$.

Without loss of generality we may assume that $s-1 \geq 15C/2$. Since $p_s$ is connected to some component $p'$ of $q'$, there exists a path $v$ in $\Gamma(G, X \cup \mathcal{H}')$ satisfying $v_- = (p_s)_-$, $v_+ = p'_+$, $\mathbf{Lab}\,(v) \in \mathcal{H}'$, $l(v) = 1$. Let $\bar{q}$ (respectively $\bar{q}'$) denote the subpath of $q$ (respectively $q'$) from $q_-$ to $(p_s)_-$ (respectively from $p'_+$ to $q'_+$). Consider a new cycle $\bar{o} = r\bar{q}v\bar{q}'$. Reasoning as before, we can find $i \in \{1,2,3\}$ such that $\bar{o}$ has $k$ isolated $E(g_i)$-components, where $k \geq \lfloor l(\bar{q})/3 \rfloor + \lfloor l(\bar{q}')/3 \rfloor - C - 1 \geq \lfloor 15C/6 \rfloor - C - 1 > C - 1 \geq 1$. Using Lemma 2.3, we get $g_i^{m_i} \in \langle \Omega \rangle$ and $k|g_i^{m_i}|_\Omega \leq K(l(\bar{q}) + l(\bar{q}') + C + 1)$. The latter inequality implies $|g_i^{m_i}|_\Omega \leq 21K$, yielding a contradiction in the usual way and proving (b) for $q$. By symmetry this property holds for $q'$ as well. □

Continuing the proof of Lemma 3.6, consider an element $x \in E_G(h)$. According to Lemma 3.1, there exists $l \in \mathbb{N}$ such that

$$xh^l x^{-1} = h^{\epsilon l}, \tag{7}$$

where $\epsilon = \pm 1$. Set $C = |x|_{X \cup \mathcal{H}'}$. After raising both sides of (7) in an integer power, we can assume that $l$ is sufficiently large to satisfy $l > 32C + 3$.

Consider a cycle $o = rqr'q'$ in $\Gamma(G, X \cup \mathcal{H}')$ satisfying $r_- = q'_+ = 1$, $r_+ = q_- = x$, $q_+ = r'_- = xh^l$, $r'_+ = q'_- = xh^l x^{-1}$, $\mathbf{Lab}\,(q) \equiv (g_1^{m_1} g_3^{m_3} g_2^{m_2})^l$, $\mathbf{Lab}\,(q') \equiv (g_1^{m_1} g_3^{m_3} g_2^{m_2})^{-\epsilon l}$, $l(q) = l(q') = 3l$, $l(r) = l(r') = C$.

Let $p_1, p_2, \ldots, p_{3l}$ and $p'_1, p'_2, \ldots, p'_{3l}$ be all components of $q$ and $q'$ respectively. Thus, $p_3, p_6, p_9, \ldots, p_{3l}$ are all $E_G(g_2)$-components of $q$. Since $l > 17C$ and $q$ is without backtracking, by Lemma 3.7, there exist indices $1 \leq s, s' \leq 3l$ such that the $E_G(g_2)$-component $p_s$ of $q$ is connected to the $E_G(g_2)$-component $p'_{s'}$ of $q'$. Without loss of generality, assume that $s \leq 3l/2$ (the other situation is symmetric). There is a path $u$ in $\Gamma(G, X \cup \mathcal{H}')$ with $u_- = (p'_{s'})_-$, $u_+ = (p_s)_+$, $\mathbf{Lab}\,(u) \in E_G(g_2)$ and $l(u) \leq 1$. We obtain a new cycle $o' = up_{s+1} \ldots p_{3l} r' p'_1 \ldots p'_{s'-1}$ in the Cayley graph $\Gamma(G, X \cup \mathcal{H}')$. Due to the choice of $s$ and $l$, the same argument as before will demonstrate that there are $E_G(g_2)$-components $p_{\bar{s}}$, $p'_{\bar{s}'}$ of $q$, $q'$ respectively, which are connected and $s < \bar{s} \leq 3l$, $1 \leq \bar{s}' < s'$ (in the case when $s > 3l/2$, the same inequalities can be achieved by simply renaming the indices correspondingly).

It is now clear that there exist $i \in \{1,2,3\}$ and connected $E_G(g_i)$-components $p_t$, $p'_{t'}$ of $q$, $q'$ ($s < t \leq 3l$, $1 \leq t' < s'$) such that $t > s$ is minimal. Let $v$ denote a path in $\Gamma(G, X \cup \mathcal{H}')$ with $v_- = (p_t)_-$, $v_+ = (p_{t'})_+$, $\mathbf{Lab}\,(v) \in E_G(g_i)$ and $l(v) \leq 1$. Consider a cycle $o''$ in $\Gamma(G, X \cup \mathcal{H}')$ defined by $o'' = up_{s+1} \ldots p_{t-1} v p'_{t'+1} \ldots p'_{s'-1}$. By part a) of Lemma 3.7, $p_{s+1}$ is a regular component of the path $p_{s+1} \ldots p_{t-1}$ in $o''$ (provided that $t-1 \geq s+1$). Note that $p_{s+1}$ can not be connected to $u$ or $v$ because $q$ is without backtracking, hence it must be connected to a component of the path $p'_{t'+1} \ldots p'_{s'-1}$. By the choice of $t$, we have

$t = s+1$ and $i = 1$. Similarly $t' = s'-1$. Thus $p_{s+1} = p_t$ and $p'_{s'-1} = p'_{t'}$ are connected $E_G(g_1)$-components of $q$ and $q'$.

In particular, we have $\epsilon = 1$. Indeed, otherwise we would have $\mathbf{Lab}\,(p_{s'-1}) \equiv g_3^{m_3}$ but $g_3^{m_3} \notin E_G(g_1)$. Therefore $x \in E_G^+(h)$ for any $x \in E_G(h)$, consequently $E_G(h) = E_G^+(h)$.

Observe that $u_- = v_+$ and $u_+ = v_-$, hence $\mathbf{Lab}\,(u)$ and $\mathbf{Lab}\,(v)^{-1}$ represent the same element $z \in E_G(g_2) \cap E_G(g_1)$. By construction, $x = h^\alpha z h^\beta$ where $\alpha = (3l - s')/3 \in \mathbb{Z}$, and $\beta = -s/3 \in \mathbb{Z}$. Thus $x \in \langle h, E_G(g_1) \cap E_G(g_2)\rangle$ and the first part of the claim 2 is proved.

Assume now that $E_G(g_j) = E_G^+(g_j)$ for $j = 1, 2$. Then $h = g_1^{m_1}(g_2 g_1^{n_1})^{m_3} g_2^{m_2}$ belongs to the centralizer of the finite subgroup $E_G(g_1) \cap E_G(g_2)$ (because of the choice of $g_1, g_2$ above). Consequently $E_G(h) = \langle h\rangle \times (E_G(g_1) \cap E_G(g_2))$. □

**Lemma 3.8.** *Let $S$ be a non-elementary subgroup of $G$ with $S^0 \neq \emptyset$. Then*

(i) *there exist non-commensurable elements $h_1, h_1' \in S^0$ with $E_G(h_1) \cap E_G(h_1') = E_G(S)$;*

(ii) *$S^0$ contains an element $h$ such that $E_G(h) = \langle h\rangle \times E_G(S)$.*

*Proof.* Choose an element $g_1 \in S^0$. By Lemma 3.1, $G$ is hyperbolic relative to the collection $\mathfrak{C} = \{H_\lambda\}_{\lambda\in\Lambda} \cup \{E_G(g_1)\}$. Since the subgroup $S$ is non-elementary, there is $a \in S \setminus E_G(g_1)$, and Lemma 3.5 provides us with an integer $n \in \mathbb{N}$ such that $g_2 = a g_1^n \in S$ is a hyperbolic element of infinite order (now, with respect to the family of peripheral subgroups $\mathfrak{C}$). In particular, $g_1$ and $g_2$ are non-commensurable and hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$.

Applying Lemma 3.6, we find $h_1 \in S^0$ (with respect to the collection of peripheral subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$) with $E_G(h_1) = E_G^+(h_1)$ such that $h_1$ is not commensurable with $g_j$, $j = 1, 2$. Hence, $g_1$ and $g_2$ stay hyperbolic after including $E_G(h_1)$ into the family of peripheral subgroups (see Lemma 3.1). This allows to construct (in the same manner) one more element $h_2 \in \langle g_1, g_2\rangle \leq S$ which is hyperbolic relative to $(\{H_\lambda\}_{\lambda\in\Lambda} \cup E_G(h_1))$ and satisfies $E_G(h_2) = E_G^+(h_2)$. In particular, $h_2$ is not commensurable with $h_1$.

We claim now that there exists $x \in S$ such that $E_G(x^{-1}h_2x) \cap E_G(h_1) = E_G(S)$. By definition, $E_G(S) \subseteq E_G(x^{-1}h_2x) \cap E_G(h_1)$. To obtain the inverse inclusion, arguing by the contrary, suppose that for each $x \in S$ we have

$$(E_G(x^{-1}h_2x) \cap E_G(h_1)) \setminus E_G(S) \neq \emptyset. \tag{8}$$

Note that if $g \in S^0$ with $E_G(g) = E_G^+(g)$, then the set of all elements of finite order in $E_G(g)$ form a finite subgroup $T(g) \leq E_G(g)$ (this is a well-known property of groups, all of whose conjugacy classes are finite). The elements $h_1$ and $h_2$ are not commensurable, therefore

$$E_G(x^{-1}h_2x) \cap E_G(h_1) = T(x^{-1}h_2x) \cap T(h_1) = x^{-1}T(h_2)x \cap T(h_1).$$

For each pair of elements $(b, a) \in D = T(h_2) \times (T(h_1) \setminus E_G(S))$ choose $x = x(b, a) \in S$ so that $x^{-1}bx = a$ if such $x$ exists; otherwise set $x(b, a) = 1$.

The assumption (8) clearly implies that $S = \bigcup_{(b,a)\in D} x(b, a) C_S(a)$, where $C_S(a)$ denotes the centralizer of $a$ in $S$. Since the set $D$ is finite, a well-know theorem of B. Neumann

[17] implies that there exists $a \in T(h_1) \setminus E_G(S)$ such that $|S : C_S(a)| < \infty$. Consequently, $a \in E_G(g)$ for every $g \in S^0$, that is, $a \in E_G(S)$, a contradiction.

Thus, $E_G(xh_2x^{-1}) \cap E_G(h_1) = E_G(S)$ for some $x \in S$. After setting $h_1' = x^{-1}h_2x \in S^0$, we see that elements $h_1$ and $h_1'$ satisfy the claim (i). Since $E_G(h_1') = x^{-1}E_G(h_2)x$, we have $E_G(h_1') = E_G^+(h_1')$. To demonstrate (ii), it remains to apply Lemma 3.6 and obtain an element $h \in \langle h_1, h_1' \rangle \leq S$ which has the desired properties. □

*Proof of Proposition 3.4.* The implication (1) ⇒ (2) is an immediate consequence of the definition. The inverse implication follows directly from the first claim of Lemma 3.8 ($S$ is non-elementary as $S^0 \neq \emptyset$ and $E_G(S) = \{1\}$). □

# 4 Proofs of the main results

The following simplification of Theorem 2.4 from [22] is the key ingredient of the proofs in the rest of the paper.

**Theorem 4.1.** *Let $U$ be a group hyperbolic relative to a collection of subgroups $\{V_\lambda\}_{\lambda \in \Lambda}$, $S$ a suitable subgroup of $U$, and $T$ a finite subset of $U$. Then there exists an epimorphism $\eta \colon U \to W$ such that:*

1. *The restriction of $\eta$ to $\bigcup_{\lambda \in \Lambda} V_\lambda$ is injective, and the group $W$ is properly relatively hyperbolic with respect to the collection $\{\eta(V_\lambda)\}_{\lambda \in \Lambda}$.*
2. *For every $t \in T$, we have $\eta(t) \in \eta(S)$.*

Let us also mention two known results we will use. The first lemma is a particular case of Theorem 1.4 from [21] (if $g \in G$ and $H \leq G$, $H^g$ denotes the conjugate $g^{-1}Hg \leq G$).

**Lemma 4.2.** *Suppose that a group $G$ is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Then*

*(a) For any $g \in G$ and any $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, the intersection $H_\lambda^g \cap H_\mu$ is finite.*

*(b) For any $\lambda \in \Lambda$ and any $g \notin H_\lambda$, the intersection $H_\lambda^g \cap H_\lambda$ is finite.*

The second result can easily be derived from Lemma 3.5.

**Lemma 4.3** (Corollary 4.5, [20])**.** *Let $G$ be an infinite properly relatively hyperbolic group. Then $G$ contains a hyperbolic element of infinite order.*

**Lemma 4.4.** *Let the group $G$ be hyperbolic with respect to the collection of peripheral subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ and let $N \lhd G$ be a finite normal subgroup. Then*

1. *If $H_\lambda$ is infinite for some $\lambda \in \Lambda$, then $N \leq H_\lambda$;*
2. *The quotient $\bar{G} = G/N$ is hyperbolic relative to the natural image of the collection $\{H_\lambda\}_{\lambda \in \Lambda}$.*

*Proof.* Let $K_\lambda$, $\lambda \in \Lambda$, be the kernel of the action of $H_\lambda$ on $N$ by conjugation. Since $N$ is finite, $K_\lambda$ has finite index in $H_\lambda$. On the other hand $K_\lambda \le H_\lambda \cap H_\lambda^g$ for every $g \in N$. If $H_\lambda$ is infinite this implies $N \le H_\lambda$ by Lemma 4.2.

To prove the second assertion, suppose that $G$ has a relatively finite presentation (2) with respect to the free product $F$ defined in (1). Denote by $\bar{X}$ and $\bar{H}_\lambda$ the natural images of $X$ and $H_\lambda$ in $\bar{G}$. In order to show that $\bar{G}$ is relatively hyperbolic, one has to consider it as a quotient of the free product $\bar{F} = (*_{\lambda\in\Lambda}\bar{H}_\lambda) * F(\bar{X})$. As $G$ is a quotient of $F$, we can choose some finite preimage $M \subset F$ of $N$. For each element $f \in M$, fix a word in $X \cup \mathcal{H}$ which represents it in $F$ and denote by $\mathcal{S}$ the (finite) set of all such words. By the universality of free products, there is a natural epimorphism $\varphi : F \to \bar{F}$ mapping $X$ onto $\bar{X}$ and each $H_\lambda$ onto $\bar{H}_\lambda$. Define the subsets $\bar{\mathcal{R}}$ and $\bar{\mathcal{S}}$ of words in $\bar{X} \cup \bar{\mathcal{H}}$ (where $\bar{\mathcal{H}} = \bigsqcup_{\lambda\in\Lambda}(\bar{H}_\lambda \setminus \{1\})$) by $\bar{\mathcal{R}} = \varphi(\mathcal{R})$ and $\bar{\mathcal{S}} = \varphi(\mathcal{S})$. Then the group $\bar{G}$ possesses the relatively finite presentation

$$\langle \bar{X},\ \{\bar{H}_\lambda\}_{\lambda\in\Lambda} \mid \bar{R} = 1,\ \bar{R} \in \bar{\mathcal{R}};\ \bar{S} = 1,\ \bar{S} \in \bar{\mathcal{S}}\rangle. \tag{9}$$

Let $\psi : F \to G$ denote the natural epimorphism and $D = \max\{\|s\| \ :\ s \in \mathcal{S}\}$. Consider any non-empty word $\bar{w}$ in the alphabet $\bar{X} \cup \bar{\mathcal{H}}$ representing the identity in $\bar{G}$. Evidently we can choose a word $w$ in $X \cup \mathcal{H}$ such that $\bar{w} =_{\bar{F}} \varphi(w)$ and $\|w\| = \|\bar{w}\|$. Since $\ker(\psi) \cdot M$ is the kernel of the induced homomorphism from $F$ to $\bar{G}$, we have $w =_F vu$ where $u \in \mathcal{S}$ and $v$ is a word in $X \cup \mathcal{H}$ satisfying $v =_G 1$ and $\|v\| \le \|w\| + D$. Since $G$ is relatively hyperbolic there is a constant $C \ge 0$ (independent of $v$) such that

$$v =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i,$$

where $R_i \in \mathcal{R}$, $f_i \in F$, and $k \le C\|v\|$. Set $\bar{R}_i = \varphi(R) \in \bar{\mathcal{R}}$, $\bar{f}_i = \varphi(f_i) \in \bar{F}$, $i = 1, 2, \ldots, k$, and $\bar{R}_{k+1} = \varphi(u) \in \bar{\mathcal{S}}$, $\bar{f}_{k+1} = 1$. Then

$$\bar{w} =_{\bar{F}} \prod_{i=1}^{k+1} \bar{f}_i^{-1} \bar{R}_i^{\pm 1} \bar{f}_i,$$

where

$$k + 1 \le C\|v\| + 1 \le C(\|w\| + D) + 1 \le C\|\bar{w}\| + CD + 1 \le (C + CD + 1)\|\bar{w}\|.$$

Thus, the relative presentation (9) satisfies a linear isoperimetric inequality with the constant $(C + CD + 1)$. □

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Observe that the quotient of $G$ by the finite normal subgroup $N = E_G(G)$ is obviously non-elementary. Hence the image of any finite $H_\lambda$ is a proper subgroup of $G/N$. On the other hand, if $H_\lambda$ is infinite, then $N \le H_\lambda \lneqq G$ by Lemma 4.4, hence its image is also proper in $G/N$. Therefore $G/N$ is properly relatively hyperbolic with respect to the collection of images of $H_\lambda$, $\lambda \in \Lambda$ (see Lemma 4.4). Lemma 3.3 implies $E_{G/N}(G/N) = \{1\}$. Thus, without loss of generality, we may assume that $E_G(G) = 1$.

It is straightforward to see that the free product $U = G * R$ is hyperbolic relative to the collection $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{R\}$ and $E_{G*R}(G) = E_G(G) = 1$. Note that $G^0$ is non-empty by Lemma 4.3. Hence $G$ is a suitable subgroup of $G * R$ by Proposition 3.4. Let $Y$ be a finite generating set of $R$. It remains to apply Theorem 4.1 to $U = G * R$, the obvious collection of peripheral subgroups, and the finite set $Y$. □

To prove Theorem 1.4 we need one more auxiliary result which was proved in the full generality in [21] (see also [6]):

**Lemma 4.5** (Theorem 2.40, [21]). *Suppose that a group $G$ is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{S_1, \ldots, S_m\}$, where $S_1, \ldots, S_m$ are hyperbolic in the ordinary (non-relative) sense. Then $G$ is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$.*

*Proof of Theorem 1.4.* Let $G_1$, $G_2$ be finitely generated groups which are properly relatively hyperbolic with respect to collections of subgroups $\{H_{1\lambda}\}_{\lambda\in\Lambda}$ and $\{H_{2\mu}\}_{\mu\in M}$ respectively. Denote by $X_i$ a finite generating set of the group $G_i$, $i = 1, 2$. As above we may assume that $E_{G_1}(G_1) = E_{G_2}(G_2) = \{1\}$. We set $G = G_1 * G_2$. Observe that $E_G(G_i) = E_{G_i}(G_i) = \{1\}$ and hence $G_i$ is suitable in $G$ for $i = 1, 2$ (by Lemma 4.3 and Proposition 3.4).

By the definition of suitable subgroups, there are two non-commensurable elements $g_1, g_2 \in G_2^0$ such that $E_G(g_1) \cap E_G(g_2) = \{1\}$. Further, by Lemma 3.1, the group $G$ is hyperbolic relative to the collection $\mathfrak{P} = \{H_{1\lambda}\}_{\lambda\in\Lambda} \cup \{H_{2\mu}\}_{\mu\in M} \cup \{E_G(g_1), E_G(g_2)\}$. We now apply Theorem 4.1 to the group $G$ with the collection of peripheral subgroups $\mathfrak{P}$, the suitable subgroup $G_1 \leq G$, and the subset $T = X_2$. The resulting group $W$ is obviously a quotient of $G_1$.

Observe that $W$ is hyperbolic relative to (the image of) the collection $\{H_{1\lambda}\}_{\lambda\in\Lambda} \cup \{H_{2\mu}\}_{\mu\in M}$ by Lemma 4.5. We would like to show that $G_2$ is a suitable subgroup of $W$ with respect to this collection. To this end we note that $\eta(g_1)$ and $\eta(g_2)$ are elements of infinite order as $\eta$ is injective on $E_G(g_1)$ and $E_G(g_2)$. Moreover, $\eta(g_1)$ and $\eta(g_2)$ are not commensurable in $W$. Indeed, otherwise, the intersection $\big(\eta(E_G(g_1))\big)^g \cap \eta(E_G(g_2))$ is infinite for some $g \in G$ that contradicts the first assertion of Lemma 4.2. Assume now that $g \in E_W(\eta(g_i))$ for some $i \in \{1, 2\}$. By the first assertion of Lemma 3.1, $\big(\eta(g_i^m)\big)^g = \eta(g_i^{\pm m})$ for some $m \neq 0$. Therefore, $\big(\eta(E_G(g_i))\big)^g \cap \eta(E_G(g_i))$ contains $\eta(g_i^m)$ and, in particular, this intersection is infinite. By the second assertion of Lemma 4.2, this means that $g \in \eta(E_G(g_i))$. Thus, $E_W(\eta(g_i)) = \eta(E_G(g_i))$. Finally, using injectivity of $\eta$ on $E_G(g_1) \cup E_G(g_2)$, we obtain

$$E_W(\eta(g_1)) \cap E_W(\eta(g_2)) = \eta(E_G(g_1)) \cap \eta(E_G(g_2)) = \eta\big(E_G(g_1) \cap E_G(g_2)\big) = \{1\}.$$

This means that the image of $G_2$ is a suitable subgroup of $W$.

Thus we may apply Theorem 4.1 again to the group $W$, the subgroup $G_2$ and the finite subset $X_1$. The resulting group $Q$ is the desired common quotient of $G_1$ and $G_2$. The last property, which claims that $Q$ can be obtained from $G_1 * G_2$ by adding only finitely many relations, follows because $G_1 * G_2$ and $G$ are hyperbolic with respect to the same family of peripheral subgroups and any relatively hyperbolic group is relatively finitely presented. □

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