RELATIVELY HYPERBOLIC GROUPS ARE C^*-SIMPLE

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Abstract. We characterize relatively hyperbolic groups whose reduced C^*-algebra is simple as those, which have no non-trivial finite normal subgroups.

1. Introduction

Let G be a countable discrete group. We denote by $\ell^2(G)$ the Hilbert space of square-summable complex-valued functions on G and by $B(\ell^2(G))$ the algebra of bounded operators on $\ell^2(G)$. The group G acts on $\ell^2(G)$ by means of the left regular representation:

$$\lambda(g)f(h) = f(g^{-1}h),\ g,h \in G, f \in \ell^2(G).$$

The *reduced C^*-algebra* $C^*_r(G)$ of G is the operator norm closure of the linear span of the set of operators $\{\lambda(g) \mid g \in G\}$ in $B(\ell^2(G))$. It has a unit element and the canonical trace $\tau : C^*_r(G) \to \mathbb{C}$, given by $\tau(1) = 1$ and $\tau(\lambda(g)) = 0$ for all $g \in G \setminus \{1\}$. This C^*-algebra reflects analytic properties of the group G. It plays an important role in non-commutative geometry and, via K-theory, in the Baum-Connes conjecture [22].

We say that the group G is *C^*-simple* if its reduced C^*-algebra is simple, that is, it has no non-trivial two-sided ideals.

In 1975 Powers established the C^*-simplicity of non-abelian free groups [21]. Later, many other examples of C^*-simple groups were found. These include non-trivial free products [20], torsion-free non-elementary Gromov hyperbolic groups [13, 15] (more generally, torsion-free non-elementary convergence groups [14]), centerless mapping class groups and outer automorphism groups of free groups [5], irreducible Coxeter groups which are neither finite nor affine [11], etc. A nice overview of C^*-simple groups can be found in [14].

The C^*-simplicity can be regarded as a strong form of non-amenability: if G is both amenable and C^*-simple then G is reduced to one element [14]. It is a classical result that the existence of free subgroups in the group implies non-amenability. In [2], Bekka, Cowling and de la Harpe introduced the following "free-like" property for finite subsets of a group:

Definition. A discrete group G is said to have *Property* $\mathrm{P}_{\mathrm{nai}}$ if for any finite subset F of $G \setminus \{1\}$ there exists an element $g_0 \in G$ of infinite order such that for each $f \in F$, the subgroup $\langle f, g_0\rangle$ of G, generated by f and g_0, is canonically isomorphic to the free product $\langle f\rangle * \langle g_0\rangle$.

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This property guarantees that $C^*_r(G)$ is simple and has a unique normalized trace [2]. Recall that a *normalized trace* on a C^*-algebra A with unit is a linear map $\sigma : A \to \mathbb{C}$ such that $\sigma(1) = 1$, $\sigma(a^*a) \geq 0$, and $\sigma(ab) = \sigma(ba)$ for all a, b in A.

In the present article our main goal is to characterize C^*-simple relatively hyperbolic groups. The class of relatively hyperbolic groups, that is, groups hyperbolic with respect to appropriate collections of subgroups, is very large. It includes Gromov hyperbolic groups and many other examples. For instance, if M is a complete Riemannian finite–volume manifold of pinched negative sectional curvature, then $\pi_1(M)$ is hyperbolic with respect to the cusp subgroups [4, 10]. More generally, if G acts isometrically and properly discontinuously on a proper hyperbolic metric space X so that the induced action of G on ∂X is geometrically finite, then G is hyperbolic relative to the collection of maximal parabolic subgroups [4]. Groups acting on $CAT(0)$ spaces with isolated flats are hyperbolic relative to the collection of flat stabilizers [16]. Algebraic examples of relatively hyperbolic groups include free products and their small cancellation quotients [19], and fully residually free groups (or Sela's limit groups) [7].

The notion of a relatively hyperbolic group was originally suggested by Gromov [12] and since then it has been investigated from different points of view [4, 10, 9, 8]. We use a general approach suggested by Osin in [19] (see the next section for details) which, when applied to finitely generated groups, is equivalent to those elaborated by Bowditch [4] and Farb (with the "bounded coset penetration" condition) [10].

Every non-elementary group G which is hyperbolic relatively to a collection of proper subgroups, or NPRH group[1] for brevity, has a *maximal finite normal subgroup* denoted by $E(G)$ [1, Lemma 3.3]. The quotient $G_1 = G/E(G)$ is again a NPRH group [1, Lemma 4.4] and, clearly, $E(G_1) = \{1\}$. The main result of this paper is the following

Theorem 1. *Let G be a non-elementary group hyperbolic relatively to a collection of proper subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. If $E(G) = \{1\}$ then G satisfies Property* $\mathrm{P_{nai}}$.

Corollary 2. *Let G be a NPRH group. Then the following are equivalent.*

(i) *The reduced C^*-algebra of G is simple;*
(ii) *The reduced C^*-algebra of G has a unique normalized trace;*
(iii) *G has infinite conjugacy classes*[2]*;*
(iv) *G does not have non-trivial finite normal subgroups.*

Proof. A discrete C^*-simple group cannot have a non-trivial amenable normal subgroup [3, Prop. 2]. A similar argument shows the same for a discrete group of which the reduced C^*-algebra has a unique normalized trace [3, Prop. 2]. Evidently, a group with infinite conjugacy classes contains no non-trivial finite normal subgroups. Therefore each of the properties (i) – (iii) implies (iv).

Suppose, now, that (iv) holds. Then Theorem 1 and the result of Bekka-Cowling-de la Harpe [2] mentioned above imply (i) and (ii). Property (iii) follows from (i) as noted in [14]. □

[1]This refers to a non-elementary properly relatively hyperbolic group. A group is *elementary* if it has a cyclic subgroup of finite index.

[2]A group G has *infinite conjugacy classes*, or, shortly, is icc, if it is infinite and if all its conjugacy classes distinct from $\{1\}$ are infinite. A group G is icc if and only if the von Neumann algebra $W^*(\Gamma)$ is a factor of type II_1 [17, Lemma 5.3.4].

Observe that a C^*-simple group is icc but the converse is not true in general as there exist amenable icc groups. However, it is an open problem to find a group G such that $C_r^*(G)$ is simple with several normalized traces, or a group such that $C_r^*(G)$ has a unique normalized trace and is not simple [3, Prop. 2].

Every non-elementary Gromov hyperbolic group is a NPRH group with respect to the family consisting of the trivial subgroup. Therefore Corollary 2 also describes all C^*-simple Gromov hyperbolic groups.

Recall that any countable group G has a maximal normal amenable subgroup $R_a(G) \lhd G$ called the *amenable radical* of G. Corollary 2 together with the existence of a maximal finite normal subgroup, imply that for any NPRH group G, the quotient $G/E(G)$ is C^*-simple. Since $R_a(G/E(G)) = R_a(G)/E(G)$ it follows (see [3, Prop. 2]) that $R_a(G) = E(G)$. We have just obtained

Corollary 3. *The amenable radical of a NPRH group G coincides with its maximal finite normal subgroup $E(G)$ and the quotient $G/E(G)$ is C^*-simple with a unique normalized trace.*

Thus, a NPRH group G is C^*-simple if and only if its amenable radical is reduced to one element. It is worth noticing that it is not yet known whether there exists a countable group G whose amenable radical is trivial but G is *not* C^*-simple [14, Question 4].

Easy examples of C^*-simple NPRH groups include non-abelian fully residually free groups G mentioned above [7]. Indeed, since non-abelian free groups satisfy Property $\mathrm{P_{nai}}$ in an obvious way, a standard argument[3] shows the same for G. This method, however, cannot be applied to prove Theorem 1, because there exist C^*-simple NPRH groups which are not fully residually hyperbolic (more generally, which are not limits of Gromov hyperbolic groups in the space of marked groups – see [6] for the definitions). As an example one can take a free product $S * \mathbb{Z}$, where S is an infinite finitely presented simple group.

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2. Relatively hyperbolic groups and their properties

Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a fixed collection of subgroups of G (called *peripheral subgroups*), X a subset of G. We say that X is a *relative generating set of* G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ if G is generated by X together with the union of all H_λ. In this situation the group G can be considered as a quotient of the free product

$$F = (*_{\lambda\in\Lambda} H_\lambda) * F(X),$$

where $F(X)$ is the free group with basis X. Let $\mathcal{R}$ be a subset of F such that the kernel of the natural epimorphism $F \to G$ is the normal closure of $\mathcal{R}$ in the group

[3]Consider a finite subset $F \subset G\setminus\{1\}$. By the assumptions, there exist a non-abelian free group H and a homomorphism $\varphi : G \to H$ such that $\varphi(F) \subset H \setminus \{1\}$. Any subgroup of a free group is free itself, hence we can suppose that φ is surjective. As H satisfies Property $\mathrm{P_{nai}}$, we can choose $h \in H$ so that for every $f \in F$, the elements $\varphi(f)$ and h freely generate a free subgroup $H' \leq H$ of rank 2. Choose an arbitrary preimage g_0 of h in G. For any $f \in F$ the restriction of φ to the subgroup $\langle f, g_0\rangle$ is an isomorphism with H', because a non-trivial element in its kernel would yield a non-trivial relation between the images $\varphi(f)$ and $\varphi(g_0) = h$. Thus, $\langle f, g_0\rangle \cong \langle f\rangle * \langle g_0\rangle$, and G satisfies Property $\mathrm{P_{nai}}$.

F; we say that G has *relative presentation*

$$\langle X,\ \{H_\lambda\}_{\lambda\in\Lambda} \mid R=1,\ R\in\mathcal{R}\rangle. \tag{1}$$

If the sets X and $\mathcal{R}$ are finite, the relative presentation (1) is said to be *finite*.

Set $\mathcal{H}=\bigsqcup_{\lambda\in\Lambda}(H_\lambda\setminus\{1\})$. A finite relative presentation (1) is said to satisfy a *linear relative isoperimetric inequality* if there exists $C>0$ such that, for every word w in the alphabet $X\cup\mathcal{H}$ (for convenience, we assume further on that $X^{-1}=X$) representing the identity in the group G, one has

$$w=_F \prod_{i=1}^{k} f_i^{-1}R_i^{\pm 1}f_i,$$

with equality in the group F, where $R_i\in\mathcal{R}$, $f_i\in F$, for $i=1,\dots,k$, and $k\le C\|w\|$, where $\|w\|$ is the length of the word w.

The group G is called *relatively hyperbolic with respect to a collection of peripheral subgroups* $\{H_\lambda\}_{\lambda\in\Lambda}$, if G admits a finite relative presentation (1) satisfying a linear relative isoperimetric inequality. This definition is independent of the choice of the finite generating set X and the finite set $\mathcal{R}$ in (1) (see [19]).

For a combinatorial path p in the Cayley graph $\Gamma(G,X\cup\mathcal{H})$ of G with respect to $X\cup\mathcal{H}$, we denote by p_-, p_+ and $\|p\|$ the *initial point*, the *end point* and the *length* correspondingly. We will write $elem(p)$ for the *element* of G represented by the label of p. Further, if Ω is a subset of G and g belongs to the subgroup $\langle\Omega\rangle\le G$ generated by Ω, then $|g|_\Omega$ will denote the *length of a shortest word* in $\Omega^{\pm 1}$, representing g.

Suppose q is a path in $\Gamma(G,X\cup\mathcal{H})$. Using the terminology of [19], a subpath p is called an H_λ*–component* (or, simply, a *component*) of q, if the label of p is a word in the alphabet $H_\lambda\setminus\{1\}$ for some $\lambda\in\Lambda$, and p is not contained in a longer subpath of q with this property.

Two components p_1,p_2 of a path q in $\Gamma(G,X\cup\mathcal{H})$ are called *connected* if they are H_λ–components for the same $\lambda\in\Lambda$ and there exists a path c in $\Gamma(G,X\cup\mathcal{H})$ connecting a vertex of p_1 to a vertex of p_2 whose label entirely consists of letters from H_λ. In algebraic terms, this means that all vertices of p_1 and p_2 belong to the same coset gH_λ for a certain $g\in G$. We can always assume c to have length at most 1, as every non-trivial element of H_λ is included in the set of generators of G. An H_λ–component p of a path q is called *isolated* if no other H_λ–component of q is connected to p.

The next statement is often useful in the study of relatively hyperbolic groups.

Lemma 4 ([19], Lemma 2.27)**.** *Suppose that a group G is hyperbolic relatively to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then there exist a finite subset $\Omega\subseteq G$ and a constant $K\ge 0$ such that the following holds.*
Let q be a cycle in $\Gamma(G,X\cup\mathcal{H})$, $p_1,\dots,p_k$ a set of isolated H_λ–components of q for some $\lambda\in\Lambda$. Denote by $g_1,\dots,g_k$ the elements of G which are represented by the labels of $p_1,\dots,p_k$ respectively. Then $g_1,\dots,g_k$ belong to the subgroup $\langle\Omega\rangle\le G$ and the word lengths of g_i's with respect to Ω satisfy the inequality

$$\sum_{i=1}^{k}|g_i|_\Omega\le K\|q\|.$$

An element $g \in G$ is called *hyperbolic* if it is not conjugate to an element of some H_λ, $\lambda \in \Lambda$. The following description of elementary subgroups in a relatively hyperbolic group was obtained by Osin.

Lemma 5 ([18], Thm. 4.3, Cor. 1.7)**.** *Let G be a group hyperbolic relatively to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Let g be a hyperbolic element of infinite order of G. Then the following conditions hold.*

(1) *The element g is contained in a unique maximal elementary subgroup $E_G(g)$ of G; moreover,*

$$E_G(g) = \{f \in G \ : \ f^{-1}g^n f = g^{\pm n} \text{ for some } n \in \mathbb{N}\}.$$

(2) *The group G is hyperbolic relative to the collection $\{H_\lambda\}_{\lambda\in\Lambda} \cup \{E_G(g)\}$.*

A significant restriction on the choice of peripheral subgroups is described in the lemma below.

Lemma 6 ([19], Thm. 1.4)**.** *Suppose that a group G is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then*

(a) *For any $g \in G$ and any $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, the intersection $H_\lambda^g \cap H_\mu$ is finite.*
(b) *For any $\lambda \in \Lambda$ and any $g \notin H_\lambda$, the intersection $H_\lambda^g \cap H_\lambda$ is finite.*

3. Proof of the main result

Throughout this section we assume that G is a non-elementary group hyperbolic relatively to a family of *proper* subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$.

Lemma 7. *Let $g \in G$ be a hyperbolic element of infinite order satisfying $E_G(g) = \langle g\rangle$. Then there exists $N \in \mathbb{N}$ such that for any $n \geq N$ and $\lambda \in \Lambda$, the subgroup $\langle H_\lambda, g^n\rangle \leq G$ is canonically isomorphic to the free product $H_\lambda * \langle g^n\rangle$.*

Proof. By Lemma 5, G is hyperbolic relatively to the collection $\{H_\lambda\}_{\lambda\in\Lambda} \cup \langle g\rangle$. Lemma 4 provides a finite subset Ω of G and a constant $K > 0$ corresponding to this new family of peripheral subgroups. Since the order of g is infinite, there exists $N \in \mathbb{N}$ such that

$$g^m \notin \{y \in \langle\Omega\rangle \ : \ |y|_\Omega \leq 8K\} \text{ provided } |m| \geq N. \tag{2}$$

Fix $n \geq N$ and $\lambda \in \Lambda$. Suppose, on the contrary, that the subgroup of G, generated by H_λ and $\langle g^n\rangle$, is not canonically isomorphic to their free product. Then there exist $k \in \mathbb{N}$, $l_1, \dots, l_k \in n\mathbb{Z} \setminus \{0\}$ and $x_1, \dots, x_k \in H_\lambda \setminus \{1\}$ such that

$$x_1 g^{l_1} x_2 g^{l_2} \cdots x_k g^{l_k} =_G 1.$$

Define $\mathcal{H}' = \bigsqcup_{\lambda\in\Lambda}(H_\lambda \setminus \{1\}) \sqcup (\langle g\rangle \setminus \{1\})$. Consider the cycle $q = p_1 p_2 \dots p_{2k}$ in the Cayley graph $\Gamma(G, X \cup \mathcal{H}')$, where $elem(p_1) = x_1$, $elem(p_2) = g^{l_1}$, $\dots$, $elem(p_{2k-1}) = x_k$, $elem(p_{2k}) = g^{l_k}$ and $\|p_i\| = 1$, $i = 1, \dots, 2k$.

Suppose, at first, that there are two $\langle g\rangle$-components p_s and p_t of q, $2 \leq s < t \leq 2k$, which are connected. Then there exists a path u between $(p_t)_-$ and $(p_s)_+$, labelled by an element from $\langle g\rangle$ (in particular, $\|u\| \leq 1$). Without loss of generality, we may assume that $(t - s)$ is minimal.

Consider the cycle $o = p_{s+1} \dots p_{t-1} u$ in $\Gamma(G, X \cup \mathcal{H}')$. If $t = s + 2$, then $elem(p_{s+1}) = elem(u^{-1}) \in H_\lambda \cap \langle g\rangle = \{1\}$ (by Lemma 6), which would contradict to the choice of q. Hence $t \geq s+4$ and $p_{s+2}, p_{s+4}, \dots, p_{t-2}$ are isolated $\langle g\rangle$-components

of o. Applying Lemma 4 to the cycle o, we achieve $elem(p_{s+2}), \dots, elem(p_{t-2}) \in \langle\Omega\rangle$ and

$$|elem(p_{s+2})|_\Omega + |elem(p_{s+4})|_\Omega + \dots + |elem(p_{t-2})|_\Omega \le K\|o\| = K(t-s).$$

Since $elem(p_{s+2}), \dots, elem(p_{t-2}) \in \langle g^n\rangle \setminus \{1\}$, we can use the formula (2) to obtain

$$8K\frac{t-s-2}{2} \le K(t-s), \quad \text{consequently} \quad 4 \le \frac{t-s}{t-s-2} \le 3.$$

A contradiction.

Therefore all $\langle g\rangle$-components of q have to be isolated. Applying Lemma 4 to the cycle q leads to a contradiction by the same argument as before. Thus, the statement is proved. □

Each NPRH group has a hyperbolic element of infinite order [18, Cor. 4.5], therefore we can use the following simplification of Lemma 3.8 from [1]:

Lemma 8. *Every NPRH group G contains a hyperbolic element g of infinite order such that $E_G(g) = \langle g\rangle \times E(G)$.*

Proof of Theorem 1. Consider an arbitrary finite subset $F = \{f_1, \dots, f_m\}$ of G. For every i, if $f_i \in F$ is an element of finite order, then we can include the finite subgroup $\langle f_i\rangle$ in the collection of peripheral subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$, preserving the relative hyperbolicity of G (for instance, by the characterization of all "hyperbolically embedded subgroups" obtained in [18, Thm. 1.5]). If $f \in F$ is a hyperbolic element of infinite order, we use Lemma 5 to include the elementary subgroup $E_G(f)$ in $\{H_\lambda\}_{\lambda\in\Lambda}$. Thus, further on, we can assume that for each $i \in \{1, \dots, m\}$ there exist $\lambda_i \in \Lambda$ and $h_i \in G$ such that

$$\langle f_i\rangle \le h_i H_{\lambda_i} h_i^{-1}. \tag{3}$$

Since $E(G) = \{1\}$ by the hypothesis, we can apply Lemma 8 to find a hyperbolic element $g \in G$ of infinite order such that $E_G(g) = \langle g\rangle$. Therefore the elements $g_i = h_i^{-1} g h_i$ satisfy the assumptions of Lemma 7, thus there exist $N_i \in \mathbb{N}$ such that the subgroup $\langle H_{\lambda_i}, g_i^{n_i}\rangle \le G$ is canonically isomorphic to the free product of H_{λ_i} and $\langle g_i^{n_i}\rangle$ for each $n_i \ge N_i$ and $i = 1, \dots, m$. Set $n = \max\{N_i \mid i = 1, \dots, m\}$. Formula (3) and the definition of g_i imply that the subgroup $\langle f_i, g^n\rangle$ of the group G is isomorphic to the free product $\langle f_i\rangle * \langle g^n\rangle$ for every $i = 1, \dots, m$. Hence G satisfies Property $\mathrm{P}_{\mathrm{nai}}$. □

References

[1] G. Arzhantseva, A. Minasyan, D. Osin, *The SQ–universality and residual properties of relatively hyperbolic groups*, preprint, 2006. arxiv:math.GR/0601590.

[2] B. Bekka and M. Cowling, P. de la Harpe, *Some groups whose reduced C^*-algebra is simple*, Inst. Hautes Études Sci. Publ. Math. No. 80, (1994), 117–134.

[3] B. Bekka and P. de la Harpe, *Groups with simple reduced C^*-algebras*, Expo. Math. 18 (2000), no. 3, 215–230.

[4] B.H. Bowditch, *Relatively hyperbolic groups,* preprint, Southampton, 1998.

[5] M.R. Bridson and P. de la Harpe, *Mapping class groups and outer automorphism groups of free groups are C^*-simple*, J. Funct. Anal. 212 (2004), no. 1, 195–205.

[6] C. Champetier and V. Guirardel, *Limit groups as limits of free groups*, Israel J. Math. 146 (2005), 1–75.

[7] F. Dahmani, *Combination of convergence groups*, Geom. Topol. 7 (2003), 933–963.

[8] T. Delzant, *Sous-groupes distingués quotients des groupes hyperboliques*, Duke Math. J. 83 (1996), no. 3, 661–682.

[9] C. Drutu and M. Sapir, *Tree graded spaces and asymptotic cones*, with appendix by D. Osin and M. Sapir, Topology 44 (2005), no. 5, 959–1058.
[10] B. Farb, *Relatively hyperbolic groups,* GAFA 8 (1998), 810–840.
[11] G. Fendler, *Simplicity of the reduced C^*-algebras of certain Coxeter groups*, Illinois J. Math. 47 (2003), no. 3, 883–897.
[12] M. Gromov, *Hyperbolic groups,* Essays in Group Theory, MSRI Series **8** (1987), (S.M. Gersten, ed.), Springer, 75–263.
[13] P. de la Harpe, *Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint*, C. R. Acad. Sci. Paris Sér. I Math. 307 (1988), no. 14, 771–774.
[14] P. de la Harpe, *On Simplicity of Reduced C^*-algebras of Groups*, preprint, 2005. arxiv:math.OA/0509450.
[15] P. de la Harpe, *Reduced C^*-algebras of discrete groups which are simple with a unique trace*, Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), 230–253, Lecture Notes in Math. 1132, Springer, Berlin, 1985.
[16] G.C. Hruska and B. Kleiner, *Hadamard spaces with isolated flats*, Geom. Topol. 9 (2005) 1501-1538.
[17] F.J. Murray and J. von Neumann, *On rings of operators. IV*, Ann. of Math. (2) 44, (1943), 716–808.
[18] D.V. Osin, *Elementary subgroups of relatively hyperbolic groups and bounded generation*, Int. J. Alg. Comp., to appear.
[19] D.V. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. 179 (2006), no. 843, vi+100 pp.
[20] W.L. Paschke and N. Salinas, *C^*-algebras associated with free products of groups*, Pacific J. Math. 82 (1979), no. 1, 211–221.
[21] R.T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators*, Duke Math. J. 42 (1975), 151–156.
[22] A. Valette, *Introduction to the Baum-Connes conjecture*, Birkhäuser Verlag, Basel, 2002.

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