On the Engulfing Property for word hyperbolic groups

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Known examples of LERF groups include free groups, surface groups and fundamental groups of certain 3-manifolds.

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Thm.(Long, 1988). Let G be a fundamental group of a closed hyp. 3-manifold. Suppose that G engulfs every f.g. $M \le G$ with $\Lambda(M) \subsetneqq S^2_\infty$. Then $\forall\, H \le_{g.f.} G$ one has $|H^* : H| < \infty$.

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Motivating the above result, Long noted that proving that a f.g. subgroup is engulfed may be much easier than showing that it is separable.

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Remark. It is easy to construct a non-q.c. sbgp. H such that $|G:H| = \infty$ and $H^* = G$.

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Cor. 2 (M., 2005). EP $\Rightarrow \forall\, H \leq_{q.c.} G$, $H^* = HQ$ where $Q = \bigcap\{K \mid |G : K| < \infty\}$.

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Thm. 2 $\Rightarrow$ $|A^* : A| < \infty$. Evidently, $B \leq A^*$, thus $|B : A| < \infty$ – a contradiction. $\square$

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Q.3. Let G be a hyperbolic mapping torus of a free group, i.e.,

$$G = HNN_{\varphi}(F(X)) \text{ where } \varphi : F(X) \hookrightarrow F(X).$$

Does G have EP ?