ON RESIDUAL PROPERTIES OF WORD HYPERBOLIC GROUPS

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Abstract. For a fixed word hyperbolic group we compare different residual properties related to quasiconvex subgroups.

1. Introduction

Any group G can be equipped with a *profinite topology* $\mathcal{PT}(G)$, whose basic open sets are cosets to normal finite index subgroups. It is easy to see that the group operations are continuous in $\mathcal{PT}(G)$. The group is residually finite if and only if the profinite topology is Hausdorff.

A subgroup $H \leq G$ is closed in $\mathcal{PT}(G)$ if and only if it is equal to an intersection of finite index subgroups; equivalently, for any element $g \notin H$ there exists a homomorphism φ from G to a finite group L such that $\varphi(g) \notin \varphi(H)$. In this case the subgroup H is called *G-separable*.

The profinite closure of a subgroup $H \leq G$, i.e., the smallest closed subset containing H, is equal to the intersection of all finite index subgroups K of G such that $H \leq K$.

A group G is said to be LERF if every finitely generated subgroup is closed in $\mathcal{PT}(G)$. The class of all LERF groups includes free groups [6], surface groups [20] and fundamental groups of certain 3-manifolds [20], [3].

Let G be a (word) hyperbolic group with a finite symmetrized generating set $\mathcal{A}$ and let $\Gamma(G, \mathcal{A})$ be the corresponding Cayley graph of G. A subset $Q \subseteq G$ is said to be *quasiconvex* if there exists a constant $\eta \geq 0$ such that for any pair of elements $u, v \in Q$ and any geodesic segment p connecting u and v, p belongs to a closed η-neighborhood of the subset Q in $\Gamma(G, \mathcal{A})$. Quasiconvex subgroups are precisely those finitely generated subgroups which are embedded in G without distortion [10, Lemma 1.6].

As it was noted in [13], in the context of word hyperbolic groups instead of studying LERF-groups it makes sense to study GFERF-groups. A hyperbolic group is called GFERF if each quasiconvex subgroup is closed in $\mathcal{PT}(G)$. Thus, within the class of hyperbolic groups, the notion of GFERF is more general than LERF: every LERF hyperbolic group is GFERF but not vice versa.

Unfortunately, it is absolutely unclear how to decide if a random word hyperbolic group is GFERF (or LERF). With this purpose, D. Long [8] and, later, G. Niblo and B. Williams [16] suggested to utilize the *engulfing* property. They say that a

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subgroup $H \leq G$ is *engulfed* if it is contained in a proper finite index subgroup of G.

The author was mainly interested in the following two theorems established by Niblo and Williams in 2002:

Theorem A. ([16, Thm. 4.1]) *Let G be a word hyperbolic group and suppose that G engulfs every finitely generated free subgroup with limit set a proper subset of the boundary of G. Then the intersection of all finite index subgroups of G is finite. If G is torsion-free then it is residually finite.*

Theorem B. ([16, Thm. 5.2]) *Let G be a word hyperbolic group which engulfs every finitely generated subgroup K such that the limit set $\Lambda(K)$ is a proper subset of the boundary of G. Then every quasiconvex subgroup of G has a finite index in its profinite closure in G.*

The main goal of this paper is to generalize Theorems A and B by weakening their assumptions and, in certain situations, strengthening their conclusions.

In a hyperbolic group G the structure of a distorted subgroup can be very complicated. Thus, the basic idea is to use assumptions which concern only quasiconvex subgroups. We prove the following results in Section 5:

Theorem 1. *Let G be a hyperbolic group with a generating set of cardinality $s \in \mathbb{N}$. Suppose that each proper free quasiconvex subgroup of rank s is engulfed in G. Then the intersection of all finite index subgroups of G is finite. If G is torsion-free then it is residually finite.*

Theorem 2. *Suppose G is a hyperbolic group which engulfs each proper quasiconvex subgroup. Let H be an arbitrary quasiconvex subgroup of G. Then H has a finite index in its profinite closure K. Moreover, $K = HQ$ where Q is the intersection of all finite index subgroups of G.*

The assumptions of Theorems 1 and 2 are, indeed, less restrictive than the assumptions of Theorems A and B, because if H is a quasiconvex subgroup of a hyperbolic group G with $|G : H| = \infty$ then the limit set $\Lambda(H)$ is a proper subset of ∂G ([22, Thm. 4], [14, Lemma 8.2]).

In the residually finite case Theorem 2 can be reformulated as follows:

Theorem 3. *Let G be a residually finite hyperbolic group where every proper quasiconvex subgroup is engulfed. Then G is GFERF.*

Combining together the claims of Theorems 1 and 3 one obtains

Corollary 1. *Let G be a torsion-free hyperbolic group where each proper quasiconvex subgroup is engulfed. Then G is GFERF.*

N. Romanovskiĭ [19] and, independently, R. Burns [2] showed that a free product of two LERF groups is again a LERF group. We suggest yet another way for constructing GFERF groups by proving a similar result for them (see Section 6):

Theorem 4. *Suppose G_1 and G_2 are GFERF hyperbolic groups. Then the free product $G = G_1 * G_2$ is also a GFERF hyperbolic group.*

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2. Preliminaries

Let G be a group with a finite symmetrized generating set $\mathcal{A}$. Naturally, this generating set gives rise to a *word length* function $|g|_G$ for every element $g \in G$. The (left-invariant) *word metric* $d : G \times G \to \mathbb{N} \cup \{0\}$ is defined by the formula $d(x, y) = |x^{-1}y|_G$ for any $x, y \in G$. This metric can be canonically extended to the Cayley graph $\Gamma(G, \mathcal{A})$ by making each edge isometric to the interval $[0, 1] \subset \mathbb{R}$.

For any three points $x, y, w \in \Gamma(G, \mathcal{A})$, the *Gromov product* of x and y with respect to w is defined as

$$(x|y)_w \stackrel{def}{=} \frac{1}{2}\Big(d(x, w) + d(y, w) - d(x, y)\Big).$$

Since the metric is left-invariant, for arbitrary $x, y, w \in G$ we have

$$(x|y)_w = (w^{-1}x|w^{-1}y)_{1_G}.$$

The group G is called (*word*) *hyperbolic* according to M. Gromov [5] if there exists $\delta \geq 0$ such that for any $x, y, z, w \in \Gamma(G, \mathcal{A})$ their Gromov products satisfy

$$(x|y)_w \geq min\{(x|z)_w, (y|z)_w\} - \delta.$$

Equivalently, G is *hyperbolic* if there exists $\delta \geq 0$ such that each geodesic triangle Δ in $\Gamma(G, \mathcal{A})$ is δ-slim, i.e., every side of Δ is contained an a δ-neighborhood of the two other sides (see [1]).

From now on we will assume that G is a hyperbolic group and the constant δ is large enough so that it satisfies both of the definitions above.

For any two points $x, y \in \Gamma(G, \mathcal{A})$ we fix a geodesic path between them and denote it by $[x, y]$. Let p be a path in the Cayley graph of G. Then p_-, p_+ will denote the startpoint and the endpoint of p, $\|p\|$ – its length; $lab(p)$, as usual, will mean the word in the alphabet $\mathcal{A}$ written on p. $elem(p) \in G$ will denote the element of the group G represented by the word $lab(p)$. If W is a word in the $\mathcal{A}$, $elem(W)$ will denote the corresponding element of the group G. For a subset $A \subset \Gamma(G, \mathcal{A})$ its closed ε-neighborhood will be denoted by $\mathcal{O}_\varepsilon(A)$.

The δ-slimness of geodesic triangles implies 2δ-slimness of all geodesic quadrangles $abcd$ in $\Gamma(G, \mathcal{A})$:

$$[a, b] \subset \mathcal{O}_{2\delta}\big([b, c] \cup [c, d] \cup [a, d]\big).$$

A path q is called (λ, c)- *quasigeodesic* if there exist $0 < \lambda \leq 1$, $c \geq 0$, such that for any subpath p of q the inequality $\lambda\|p\| - c \leq d(p_-, p_+)$ holds. A word W is said to be (λ, c)-quasigeodesic if some (equivalently, every) path q in $\Gamma(G, \mathcal{A})$ labelled by W is (λ, c)-quasigeodesic.

Lemma 2.1. ([4, 5.6,5.11],[1, 3.3]) *There is a constant $\nu = \nu(\delta, \lambda, c)$ such that for any (λ, c)-quasigeodesic path p in $\Gamma(G, \mathcal{A})$ and a geodesic q with $p_- = q_-$, $p_+ = q_+$, one has $p \subset \mathcal{O}_\nu(q)$ and $q \subset \mathcal{O}_\nu(p)$.*

Lemma 2.2. ([14, Lemma 4.1]) *Consider a geodesic quadrangle $x_1x_2x_3x_4$ in the Cayley graph $\Gamma(G, \mathcal{A})$ with $d(x_2, x_3) > d(x_1, x_2) + d(x_3, x_4)$. Then there are points $u, v \in [x_2, x_3]$ such that $d(x_2, u) \leq d(x_1, x_2)$, $d(v, x_3) \leq d(x_3, x_4)$ and the geodesic subsegment $[u, v]$ of $[x_2, x_3]$ lies 2δ-close to the side $[x_1, x_4]$.*

If $x, g \in G$, we define $x^g = gxg^{-1}$. For a subset A of the group G, $A^g = gAg^{-1}$, and the notation A^G will be used to denote the subset $\{gag^{-1} \mid a \in A, g \in G\} \subset G$.

Remark 2.1. [10, Remark 2.2] Let $Q \subseteq G$ be quasiconvex, $g \in G$. Then the subsets gQ, Qg and gQg^{-1} are quasiconvex as well.

Thus, a conjugate of a quasiconvex subgroup in a hyperbolic group is quasiconvex itself. Another important property of hyperbolic groups states that any cyclic subgroup is quasiconvex (see [1], for instance).

In this paper we will also use the concept of *Gromov boundary* of a hyperbolic group G (for a detailed theory the reader is referred to the corresponding chapters in [4] or [1]).) A sequence $(x_i)_{i\in\mathbb{N}}$ of elements of the group G is said to be *converging to infinity* if

$$\lim_{i,j\to\infty}(x_i|x_j)_{1_G} = \infty.$$

Two sequences $(x_i)_{i\in\mathbb{N}}, (y_j)_{i\in\mathbb{N}}$ converging to infinity are said to be equivalent if

$$\lim_{i\to\infty}(x_i|y_i)_{1_G} = \infty.$$

The points of the boundary ∂G are identified with the equivalence classes of sequences converging to infinity. It is easy to see that this definition does not depend on the choice of a basepoint: instead of 1_G one could use any fixed point p of $\Gamma(G,\mathcal{A})$ (see [1]). If α is the equivalence class of $(x_i)_{i\in\mathbb{N}}$, we will write $\lim\limits_{i\to\infty} x_i = \alpha$.

The space ∂G can be topologized so that it becomes compact, Hausdorff and metrizable (see [1],[4]).

Left multiplication by an element of the group induces a homeomorphic action of G on its boundary: for any $g \in G$ and $[(x_i)_{i\in\mathbb{N}}] \in \partial G$ set

$$g \circ [(x_i)_{i\in\mathbb{N}}] \stackrel{def}{=} [(gx_i)_{i\in\mathbb{N}}] \in \partial G.$$

If $g \in G$ is an element of infinite order then the sequences $(g^i)_{i\in\mathbb{N}}$ and $(g^{-i})_{i\in\mathbb{N}}$ converge to infinity and we will use the notation

$$\lim_{i\to\infty} g^i = g^{\infty} \in \partial G,\ \lim_{i\to\infty} g^{-i} = g^{-\infty} \in \partial G.$$

For a subset $A \subseteq G$ the *limit set* $\Lambda(A)$ of A is the collection of the points $\alpha \in \partial G$ that are limits of sequences (converging to infinity) from A.

Let us recall an auxiliary binary relation defined between subsets of an arbitrary group G in [14]: suppose $A, B \subseteq G$, we will write $A \preceq B$ if and only if there exist elements $x_1, \dots, x_n \in G$ such that

$$A \subseteq Bx_1 \cup Bx_2 \cup \dots \cup Bx_n.$$

It is not difficult to see that the relation "$\preceq$" is transitive and for any $g \in G$, $A \preceq B$ implies $gA \preceq gB$.

Lemma 2.3. ([14, Lemma 2.1]) *Let A, B be subgroups of G. Then $A \preceq B$ if and only if the index $|A : (A \cap B)|$ is finite.*

The basic properties of limit sets are described in the following statement:

Lemma 2.4. ([7],[22],[14, Lemma 6.2]) *Suppose A, B are arbitrary subsets of G, $g \in G$. Then*

(a) $\Lambda(A) = \emptyset$ *if and only if A is finite;*
(b) $\Lambda(A)$ *is a closed subset of the boundary ∂G;*
(c) $\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B)$;
(d) $\Lambda(Ag) = \Lambda(A)$, $g \circ \Lambda(A) = \Lambda(gA)$;
(e) *if $A \preceq B$ then* $\Lambda(A) \subseteq \Lambda(B)$.

The following property of limit sets of quasiconvex subgroups was first proved by E. Swenson:

Lemma 2.5. ([22, Thm. 8], [11, Lemma 9.1]) *Let A, B be quasiconvex subgroups of a hyperbolic group G. Then $\Lambda(A) \cap \Lambda(B) = \Lambda(A \cap B)$ in ∂G.*

3. Auxiliary Facts

Lemma 3.1. *Assume H is an η-quasiconvex subgroup of a δ-hyperbolic group G, X is a word over $\mathcal{A}$ representing an element of infinite order in G, $0 < \lambda \leq 1$ and $c \geq 0$. Let $\nu = \nu(\delta, \lambda, c)$ be the constant given by Lemma 2.1. There exists a number $N = N(\delta, \eta, \nu, G) \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ the following property holds.*

If a word $W \equiv U_1 X^n U_2$ is (λ, c)-quasigeodesic and satisfies $\|U_1\|, \|U_2\| > (m + \nu + c)/\lambda$, $n \geq N$ and $elem(V_1 W V_2) \in H$ for some words V_1, V_2 with $\|V_1\|, \|V_2\| \leq m$, then there exist $k \in \mathbb{N}$ and $a \in G$ such that $elem(X^k) \in H^a$ and $|a|_G \leq 2\delta + \nu + \eta$.

Proof. Consider a path q in $\Gamma(G, \mathcal{A})$ starting at 1_G and labelled by $V_1 W V_2$. By our assumptions, $q_+ \in H$. Let p and r be its (λ, c)-quasigeodesic subpaths with $lab(p) \equiv W$ and $lab(r) \equiv X^n$ respectively. Choose an arbitrary *phase* vertex $u \in r$ such that the subpath of r from r_- to u is labelled by some power of X.

According to Lemma 2.1 there is $v \in [p_-, p_+]$ satisfying $d(u, v) \leq \nu$. Using the assumptions and the triangle inequality one can achieve

$$d(p_-, v) \geq d(p_-, u) - d(u, v) \geq \lambda \|U_1\| - c - \nu > m, \text{ and}$$

$$d(p_+, v) \geq d(p_+, u) - d(u, v) \geq \lambda \|U_2\| - c - \nu > m.$$

Hence, by Lemma 2.2, $v \in \mathcal{O}_{2\delta}([1_G, q_+])$.

Thus,

$$u \in \mathcal{O}_{2\delta+\nu}([1_G, q_+]) \subset \mathcal{O}_{2\delta+\nu+\eta}(H),$$

i.e., there is an element $a = a(u) \in G$ such that $|a|_G \leq 2\delta + \nu + \eta$ and $u \in Ha$.

Now, since the alphabet $\mathcal{A}$ is finite, there are only finitely many elements in G having length at most $(2\delta + \nu + \eta)$. Hence, if n is large enough, there should be two different phase vertices $u_1, u_2 \in r$ with $a(u_1) = a(u_2) = a$. By the construction,

$$u_1^{-1} u_2 = elem(X^k) \in a^{-1} H a$$

for some positive integer k (X^k is a label of the segment of r from u_1 to u_2, provided these points are chosen in a correct order). Q.e.d. □

Lemma 3.2. *Assume G is a hyperbolic group and $H \leq G$ is a quasiconvex subgroup. If $g \in G$ and $gH \preceq H$, then $H \preceq gH$.*

Proof. If H is finite, the statement is trivial. Our assumptions immediately imply

$$g^{k-1} H \preceq g^{k-2} H \preceq \cdots \preceq gH \preceq H$$

for all $k \in \mathbb{N}$. Hence,

$$g^{k-1} H \preceq H. \tag{1}$$

If g has order $k \in \mathbb{N}$ in the group G, then to achieve the desired result it is enough to multiply both sides of the above formula by g.

Thus, we can further assume that H is infinite and the element g has infinite order. Therefore, H has at least one limit point $\alpha \in \Lambda(H)$. Observe that (1) implies

$$g^n \circ \Lambda(H) = \Lambda(g^n H) \subseteq \Lambda(H),$$

thus $g^n \circ \alpha \in \Lambda(H)$ for all $n \in \mathbb{N}$.

If $\alpha \neq g^{-\infty}$ in ∂G, it is well known (see, for instance, [4, 8.16]) that the sequence $(g^n \circ \alpha)_{n\in\mathbb{N}}$ converges to g^{∞}. Since $\Lambda(H)$ is a closed subset of the boundary ∂G, in either case we achieve

$$\Lambda(\langle g\rangle_\infty) = \{g^\infty, g^{-\infty}\} \cap \Lambda(H) \neq \emptyset.$$

By Lemma 2.5, the latter implies $g^k \in H$ for some $k \in \mathbb{N}$. Combining this fact with (1) we get $H = g^k H \preceq gH$, which concludes the proof. □

The previous lemma has an interesting corollary:

Lemma 3.3. *Suppose H, K are subgroups of a hyperbolic group G such that $H \le K$, $|K : H| = \infty$ and H is quasiconvex. Then $|K : (K \cap H^g)| = \infty$ for any $g \in G$.*

Proof. If $|K : (K \cap H^g)| < \infty$ for some $g \in G$ then $H \preceq K \preceq gHg^{-1} \preceq gH$ according to Lemma 2.3. Consequently $g^{-1}H \preceq H$. Hence $H \preceq g^{-1}H$ by Lemma 3.2, implying $gHg^{-1} \preceq gH \preceq H$. But the latter leads to $K \preceq H$ which contradicts the condition $|K : H| = \infty$ (see Lemma 2.3). □

If G is a hyperbolic group, each element $g \in G$ of infinite order belongs to a unique *maximal elementary subgroup* $E(g)$. By [17, Lemmas 1.16, 1.17] it has the following description:

$$(2)\quad E(g) = \big\{x \in G \mid xg^kx^{-1} = g^l \text{ for some } k, l \in \mathbb{Z}\backslash\{0\}\big\} = \big\{x \in G \mid xg^nx^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{N}\big\}.$$

Note that the subgroup $E^+(g) \stackrel{def}{=} \big\{\ x \in G \mid xg^nx^{-1} = g^n \text{ for some } n \in \mathbb{N}\ \big\}$ has index at most 2 in $E(g)$.

Let $W_1, W_2, \dots, W_l$ be words in $\mathcal{A}$ representing elements $g_1, g_2, \dots, g_l$ of infinite order, where $E(g_i) \neq E(g_j)$ for $i \neq j$. The following lemma will be useful:

Lemma 3.4. ([17, Lemma 2.3]) *There exist constants $\lambda = \lambda(W_1, W_2, \dots, W_l) > 0$, $c = c(W_1, W_2, \dots, W_l) \ge 0$ and $N = N(W_1, W_2, \dots, W_l) > 0$ such that any path p in the Cayley graph $\Gamma(G, \mathcal{A})$ with label $W_{i_1}^{m_1} W_{i_2}^{m_2} \dots W_{i_s}^{m_s}$ is (λ, c)-quasigeodesic if $i_k \neq i_{k+1}$ for $k = 1, 2, \dots, s-1$, and $|m_k| > N$ for $k = 2, 3, \dots, s-1$ (each i_k belongs to $\{1, \dots, l\}$).*

For a subgroup H of G denote by H^0 the set of elements of infinite order in H; if $A \subseteq G$, the subgroup $C_H(A) = \{h \in H \mid g^h = g, \ \forall\ g \in A\}$ is the *centralizer* of A in H.

Set $E(H) = \bigcap_{x\in H^0} E(x)$. If H is a non-elementary subgroup of G, then $E(H)$ is the unique maximal finite subgroup of G normalized by H ([17, Prop. 1]).

If $g \in G^0$, $T(g)$ will be used to denote the set of elements of finite order in the subgroup $E(g)$.

Let G be a hyperbolic group and H be its non-elementary subgroup. Recalling the definition from [17] (and using the terminology from [12]), an element $g \in H^0$ will be called *H-suitable* if $E(H) = T(g)$ and

$$E(g) = E^+(g) = C_G(g) = T(g) \times \langle g\rangle.$$

In particular, if the element g is H-suitable then $g \in C_H\big(E(H)\big)$.

Two elements $g, h \in G$ of infinite order are called *commensurable* if $g^k = ah^la^{-1}$ for some non-zero integers k, l and some $a \in G$. The following important statement was proved by A. Ol'shanskii in 1993:

Lemma 3.5. ([17, Lemma 3.8]) *Every non-elementary subgroup H of a hyperbolic group G contains an infinite set of pairwise non-commensurable H-suitable elements.*

Suitable elements can be modified in a natural way:

Lemma 3.6. ([12, Lemma 4.3]) *Let H be a non-elementary subgroup of a hyperbolic group G, and g be an H-suitable element. If $y \in C_H\big(E(H)\big)\backslash E(g)$ then there exists $N \in \mathbb{N}$ such that the element yg^n has infinite order in H and is H-suitable for every $n \geq N$.*

In [14] the author studied properties of quasiconvex subgroups of infinite index and showed

Lemma 3.7. ([14, Prop. 1]) *Suppose H is a quasiconvex subgroup of a hyperbolic group G and K is any subgroup of G that satisfies $|K : (K \cap H^g)| = \infty$ for all $g \in G$. Then there exists an element $x \in K$ having infinite order, such that $\langle x\rangle_\infty \cap H^G = \{1_G\}$.*

Later we will utilize a stronger fact:

Lemma 3.8. *Assume H, K are subgroups of a δ-hyperbolic group G, H is η-quasiconvex, K is non-elementary and $|K : (K \cap H^g)| = \infty$ for every $g \in G$. Then there exists a K-suitable element $y \in K$ such that $\langle y\rangle_\infty \cap H^G = \{1_G\}$.*

Proof. Set $K' = C_K\big(E(K)\big)$. Since $E(K)$ is a finite subgroup normalized by K, we have $|K : K'| < \infty$. Therefore $|K' : (K' \cap H^g)| = \infty$ for all $g \in G$. Applying Lemma 3.7, we can find an element of infinite order $x \in K'$ such that $\langle x\rangle_\infty \cap H^G = \{1_G\}$. By Lemma 3.5 there is a K-suitable element $z \in K$ which is non-commensurable with x, hence $\langle x\rangle_\infty \cap E(z) = \{1_G\}$.

Choose some words X and Z in the alphabet $\mathcal{A}$ representing x and z respectively. Then one is able to find the numbers $\lambda = \lambda(X, Z)$, $c = c(X, Z)$ and $N_1 = N_1(X, Z)$ from the claim of Lemma 3.4.

Define $\nu = \nu(\delta, \lambda, c)$ and $N_2 = N_2(\delta, \eta, \nu, G)$ as in Lemmas 2.1 and 3.1. Denote $N = \max\{N_1, N_2\}$ and apply Lemma 3.6 to obtain $n \geq N_1$ such that the element $y = x^N z^n \in K$ is K-suitable.

It remains to check that $\langle y\rangle_\infty \cap H^G = \{1_G\}$. Assume the contrary, i.e., there exist $t \in \mathbb{N}$ and $g \in G$ such that $y^t \in H^g$. Then for each $l \in \mathbb{N}$, the element $y^{tl} \in H^g$ will be represented by the (λ, c)-quasigeodesic word $W \equiv (X^N Z^n)^{tl}$. And if the number l is chosen sufficiently large (compared to $m = |g^{-1}|_G = |g|_G$), it should be possible to find a subword of the form X^N in the "middle" of W which satisfies all the assumptions of Lemma 3.1. Hence, $x^k = elem(X^k) \in H^G$ for some $k \in \mathbb{N}$. The latter contradicts to the construction of x. Thus the lemma is proved. □

As usual, let G be a δ-hyperbolic group and $H-$ its η-quasiconvex subgroup.

Lemma 3.9. *Suppose the elements $x_1, x_2 \in G$ have infinite order, $E(x_1) \neq E(x_2)$, and for each $i = 1, 2$, satisfy $\langle x_i\rangle_\infty \cap H^G = \{1_G\}$. Then there exists a number $N \in \mathbb{N}$ such that for any $m, n \geq N$ the elements x_1^m, x_2^n freely generate a free subgroup F of rank 2 in G. Moreover, F is quasiconvex and $F \cap H^G = \{1_G\}$.*

Proof. Choose some words X_1 and X_2 in the alphabet $\mathcal{A}$ representing the elements x_1 and x_2. Apply Lemma 3.4 to find the corresponding $\lambda = \lambda(X_1, X_2)$, $c = c(X_1, X_2)$ and $N_1 = N_1(X_1, X_2)$. Then one can find the constant $\nu = \nu(\delta, \lambda, c)$

from the claim of Lemma 2.1 and define $N_2 = N_2(\delta, \eta, \nu, G)$ in accordance with Lemma 3.1.

Set $N = \max\{N_1, N_2, \lfloor c/\lambda \rfloor + 1\}$. Consider arbitrary integers $m, n \geq N$ and the subgroup $F = \langle x_1^m, x_2^n \rangle \leq G$. By Lemma 3.4 any non-empty (freely) reduced word W in the generators $\{X_1^m, X_2^n\}$ is (λ, c)-quasigeodesic. Hence

$$|elem(W)|_G \geq \lambda \|W\| - c \geq \lambda N - c > 0.$$

Consequently, $elem(W) \neq 1_G$ in G, implying that F is free with the free generating set $\{x_1^m, x_2^n\}$. By the construction of ν, F will be ε-quasiconvex where

$$\varepsilon = \nu + \frac{1}{2} \max\{m\|X_1\|, n\|X_2\|\}.$$

Consider a non-empty cyclically reduced word W in the generators $\{X_1^m, X_2^n\}$. For establishing the last claim, it is sufficient to demonstrate that $elem(W) \notin H^G$. Arguing as in Lemma 3.8, suppose $elem(W) \in H^g$ for some $g \in G$. Then $(elem(W))^l \in H^g$ for every $l \in \mathbb{N}$. By choosing l sufficiently large and applying Lemma 3.1 one will obtain a contradiction with the assumption $\langle x_i \rangle_\infty \cap H^G = \{1_G\}$, $i = 1, 2$, as in Lemma 3.8. Therefore $F \cap H^G = \{1_G\}$. Q.e.d. $\square$

Corollary 2. *With the assumptions of Lemma 3.8, K has a free subgroup F of rank 2 which is quasiconvex in G, $E(F) = E(K)$ and $F \cap H^G = \{1_G\}$.*

Proof. Choose a K-suitable element $x_1 \in K$ according to Lemma 3.8. Since K is non-elementary, there exists $y \in K \backslash E(x_1)$. Therefore, $x_2 \overset{def}{=} y x_1 y^{-1} \in K^0$ and $E(x_2) \neq E(x_1)$ (as it can be seen from (2)). By the construction, $\langle x_i \rangle_\infty \cap H^G = \{1_G\}$, $i = 1, 2$. Hence the subgroup F can be found by applying Lemma 3.9. Evidently $E(K) \leq E(F)$, and $E(F) \subseteq T(x_1) = E(K)$. Thus $E(F) = E(K)$. $\square$

4. Free products of quasiconvex subgroups

Below we will assume that G is a δ-hyperbolic group generated by a finite set $\mathcal{A}$. First let us recall some properties of the hyperbolic boundary.

Lemma 4.1. ([12, Lemma 2.14]) *Suppose A and B are arbitrary subsets of G and $\Lambda(A) \cap \Lambda(B) = \emptyset$. Then $\sup_{a \in A,\ b \in B} \{(a|b)_{1_G}\} < \infty$.*

Remark 4.1. Suppose $g, x \in G$ and g has infinite order. If $(x \circ \{g^{\pm\infty}\}) \cap \{g^{\pm\infty}\} \neq \emptyset$ in the boundary ∂G, then $x \in E(g)$. If $E(g) = E^+(g)$ then $g^\infty \notin G \circ \{g^{-\infty}\}$.

Note that $x \circ \{g^{\pm\infty}\} = \{(xgx^{-1})^{\pm\infty}\} = \Lambda(\langle xgx^{-1} \rangle) \subset \partial G$. Since any cyclic subgroup in a hyperbolic group is quasiconvex, we can apply Lemma 2.5 to show that $\langle g \rangle \cap \langle xgx^{-1} \rangle \neq \{1_G\}$. Hence $x \in E(g)$.
Let $E(g) = E^+(g)$ and suppose that $g^\infty = x \circ g^{-\infty}$ for some $x \in G$. Then, as we showed above, $x \in E(g) = E^+(g)$. Hence

$$x \circ g^{-\infty} = \lim_{n \to -\infty} (xgx^{-1})^n = \lim_{n \to -\infty} g^n = g^{-\infty}.$$

Thus we achieve a contradiction with the inequality $g^\infty \neq g^{-\infty}$.

Lemma 4.2. *Let $g, x \in G$ where g has infinite order and $E(g) = E^+(g)$. Then there is $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$ the element $xg^n \in G$ has infinite order.*

Proof. If $x \notin E(g)$ then the claim follows by [11, Lemma 9.14].

So, assume $x \in E(g)$. Since $E(g) = E^+(g)$, the center of $E(g)$ has finite index in it, thus $E(g)$ is an FC-group. By B.H. Neumann's Theorem [15] the elements of finite order form a subgroup $T(g) \le E(g)$. Therefore the cardinality of the intersection $\{xg^k \mid k \in \mathbb{Z}\} \cap T(g)$ can be at most 1. Thus $xg^n \notin T(g)$ for each sufficiently large n. □

The main result of this paper is based on the following statement concerning broken lines in a δ-hyperbolic metric space:

Lemma 4.3. ([18, Lemma 21], [14, Lemma 3.5]) *Let $p = [y_0, y_1, \dots, y_n]$ be a broken line in $\Gamma(G, \mathcal{A})$ such that $||[y_{i-1}, y_i]|| > C_1$ $\forall$ $i = 1, \dots, n$, and $(y_{i-1}|y_{i+1})_{y_i} < C_0$ $\forall$ $i = 1, \dots, n-1$, where $C_0 \ge 14\delta$, $C_1 > 12(C_0 + \delta)$. Then the geodesic segment $[y_0, y_n]$ is contained in the closed 14δ-neighborhood of p and $\|[y_0, y_n]\| \ge \|p\|/2$.*

Suppose a, b, c and d are arbitrary points in $\Gamma(G, \mathcal{A})$. Considering the geodesic triangle with the vertices $1_G, a$ and ab, one can observe that

$$(a|ab)_{1_G} = |a|_G - (1_G|ab)_a = |a|_G - (a^{-1}|b)_{1_G}.$$

Now, since $\Gamma(G, \mathcal{A})$ is δ-hyperbolic,

$$(3)\quad (a|c)_{1_G} \ge \min\big\{(a|ab)_{1_G}, (ab|c)_{1_G}\big\} - \delta = \min\big\{|a|_G - (a^{-1}|b)_{1_G}, (ab|c)_{1_G}\big\} - \delta.$$

Replacing c with cd in the above formula, we get

$$(4)\qquad (a|cd)_{1_G} \ge \min\big\{|a|_G - (a^{-1}|b)_{1_G}, (ab|cd)_{1_G}\big\} - \delta.$$

The Gromov product is symmetric, therefore we are able to combine formulas (3) and (4) to achieve

$$(5)\quad (a|c)_{1_G} \ge \min\big\{|c|_G - (c^{-1}|d)_{1_G}, (a|cd)_{1_G}\big\} - \delta \ge \min\big\{|a|_G - (a^{-1}|b)_{1_G}, |c|_G - (c^{-1}|d)_{1_G}, (ab|cd)_{1_G}\big\} - 2\delta.$$

Theorem 5. *Consider some elements $g_1, x_1, g_2, x_2, \dots, g_s, x_s \in G$ and an η-quasiconvex subgroup $H \le G$. Suppose the following three conditions are satisfied:*

- *$g_1, \dots, g_s$ have infinite order and are pairwise non-commensurable;*
- *$E(g_i) = E^+(g_i)$ for each $i = 1, 2, \dots, s$;*
- *$E(g_i) \cap H = E(g_i) \cap x_i^{-1} H x_i = \{1_G\}$ for each $i = 1, 2, \dots, s$.*

Then there exists a number $N \in \mathbb{N}$ such that for every $n \ge N$ the elements $x_i g_i^n \in G$, $i = 1, 2 \dots, s$, have infinite order, and the subgroup

$$M \stackrel{def}{=} \langle H, x_1 g_1^n, \dots, x_s g_s^n\rangle \le G$$

is quasiconvex in G and isomorphic (in the canonical way) to the free product $H * \langle x_1 g_1^n\rangle * \dots * \langle x_s g_s^n\rangle$.

Proof. Choose arbitrary elements $w_1, w_2 \in M$ and define $w = w_1^{-1} w_2 \in M$. Then this element has a presentation

$$(6)\qquad w = h_0 (x_{i_1} g_{i_1}^n)^{\epsilon_1} h_1 (x_{i_2} g_{i_2}^n)^{\epsilon_2} \cdots h_{l-1} (x_{i_l} g_{i_l}^n)^{\epsilon_l} h_l,$$

where $h_j \in H$, $i_j \in \{1, \dots, s\}$, $\epsilon_j \in \{1, -1\}$, $j = 1, 2, \dots, l$, $l \in \mathbb{N} \cup \{0\}$.

Moreover, we can assume that the presentation (6) is reduced in the following sense: if $1 \le j \le l-1$, $i_j = i_{j+1}$ and $\epsilon_{j+1} = -\epsilon_j$ then $h_j \ne 1_G$.

Consider a geodesic broken line $[y_0, y_1, \ldots, y_{l+1}]$ in $\Gamma(G,\mathcal{A})$ with $y_0 = w_1$ and $elem([y_k, y_{k+1}]) = h_k(x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}$, $k = 0, 1, \ldots, l-1$, $elem([y_l, y_{l+1}]) = h_l$. Therefore $elem([y_0, y_{l+1}]) = w$ and $y_{l+1} = w_2$.

Now we are going to find upper bounds for the Gromov products

$$(y_{k-1}|y_{k+1})_{y_k} = (y_k^{-1}y_{k-1}|y_k^{-1}y_{k+1})_{1_G},\ k = 1, \ldots, l.$$

By the assumptions of the theorem, Lemma 2.5 implies that

$$x_i \circ g_i^{\infty} = (x_i g_i x_i^{-1})^{\infty} \notin \Lambda(H) \text{ and } g_i^{-\infty} \notin \Lambda(H),\ i = 1, \ldots, s.$$

Since

$$\Lambda\left(\{x_i g_i^m, g_i^{-m} x_i^{-1} \mid m \in \mathbb{N}, 1 \le i \le s\}\right) = \{x_i \circ g_i^{\infty}, g_i^{-\infty} \mid 1 \le i \le s\} \subset \partial G,$$

we are able to apply Lemma 4.1 to define

$$\alpha \stackrel{def}{=} \max\left\{(h|x_i g_i^m)_{1_G}, (h|g_i^{-m} x_i^{-1})_{1_G} \;\middle|\; h \in H, 1 \le i \le s, m \in \mathbb{N}\right\} < \infty.$$

Similarly, since g_i and g_j are non-commensurable if $i \neq j$ and $E(g_i) = E^+(g_i)$, we have (according to Lemma 2.5 and Remark 4.1)

$$G \circ \{g_i^{\pm\infty}\} \cap G \circ \{g_j^{\pm\infty}\} = \emptyset,\ G \circ \{g_i^{\infty}\} \cap G \circ \{g_i^{-\infty}\} = \emptyset,\ i, j \in \{1, \ldots, s\}, i \neq j.$$

Hence, the following numbers are also finite:

$$\beta_1 \stackrel{def}{=} \max\Big\{\big((x_i g_i^m)^{-1}|h x_j g_j^t\big)_{1_G},\ \big(x_i g_i^m|h(x_j g_j^t)^{-1}\big)_{1_G} \Big| \\ h \in H,\ |h|_G \le 2\alpha + 2\delta,\ 1 \le i, j \le s,\ m, t \in \mathbb{N}\Big\},$$

$$\beta_2 \stackrel{def}{=} \max\Big\{\big(x_i g_i^m|h x_j g_j^t\big)_{1_G},\ \big((x_i g_i^m)^{-1}|h(x_j g_j^t)^{-1}\big)_{1_G} \Big| \\ h \in H,\ |h|_G \le 2\alpha + 2\delta,\ 1 \le i, j \le s,\ i \neq j,\ m, t \in \mathbb{N}\Big\}.$$

Note that if $h \in H \backslash \{1_G\}$ then, according to our assumptions, $x_i^{-1} h x_i \notin E(g_i)$. Therefore, $\{g_i^{\pm\infty}\} \cap (x_i^{-1} h x_i) \circ \{g_i^{\pm\infty}\} = \emptyset$ (Remark 4.1), implying $x_i \circ g_i^{\infty} \neq (h x_i) \circ g_i^{\infty}$, $i = 1, \ldots, s$. We can also use Remark 4.1 to show that $g_i^{-\infty} \neq h \circ g_i^{-\infty}$ for each i. Consequently, by Lemma 4.1,

$$\beta_3 \stackrel{def}{=} \max\Big\{\big(x_i g_i^m|h x_i g_i^t\big)_{1_G},\ \big((x_i g_i^m)^{-1}|h(x_i g_i^t)^{-1}\big)_{1_G} \Big| \\ h \in H,\ |h|_G \le 2\alpha + 2\delta,\ h \neq 1_G,\ 1 \le i \le s,\ m, t \in \mathbb{N}\Big\} < \infty.$$

Finally, define $\beta = \max\{\beta_1, \beta_2, \beta_3\} < \infty$,

$$C_0 = \max\{\alpha + 2\delta, \beta + \delta, 14\delta\} + 1 \text{ and } C_1 = 12(C_0 + \delta) + 1. \tag{7}$$

Since each g_i, $i = 1, \ldots, s$, has infinite order in G there exists $N \in \mathbb{N}$ such that for any $i \in \{1, \ldots, s\}$ one has

$$|g_i^n|_G > \max\{\alpha, \beta, 2C_1\} + \alpha + |x_i|_G + 2\delta,\ \forall\, n \ge N. \tag{8}$$

By Lemma 4.2, without loss of generality, we can assume that the order of $x_i g_i^n$, $i = 1, 2, \ldots, s$, is infinite for every $n \ge N$. Fix an integer $n \ge N$ and choose any $k \in \{1, \ldots, l-1\}$. Then

$$(y_{k-1}|y_{k+1})_{y_k} = \big((x_{i_k} g_{i_k}^n)^{-\epsilon_k} h_{k-1}^{-1} \big| h_k (x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}\big)_{1_G}.$$

To simplify the notation, let us denote $a = (x_{i_k} g_{i_k}^n)^{-\epsilon_k}$, $b = h_{k-1}^{-1}$, $c = h_k$ and $d = (x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}$. By construction

$$(a|c)_{1_G}, (a^{-1}|b)_{1_G}, (c^{-1}|d)_{1_G} \leq \alpha. \tag{9}$$

Now we need to consider two separate cases.

Case 1: $|h_k|_G = |c|_G \leq 2\alpha + 2\delta$. Then, due to the definition of the number β, the inequality $(a|cd)_{1_G} \leq \beta$ holds. Therefore, applying formulas (4) and (9) one obtains

$$\beta \geq (a|cd)_{1_G} \geq \min\left\{|a|_G - (a^{-1}|b)_{1_G}, (ab|cd)_{1_G}\right\} - \delta \geq \\ \min\left\{|g_{i_k}^n|_G - |x_{i_k}|_G - \alpha, (ab|cd)_{1_G}\right\} - \delta.$$

By (8), $|g_{i_k}^n|_G - |x_{i_k}|_G - \alpha > \beta + \delta$, hence the above inequality implies

$$(y_{k-1}|y_{k+1})_{y_k} = (ab|cd)_{1_G} \leq \beta + \delta < C_0.$$

Case 2: $|h_k|_G = |c|_G > 2\alpha + 2\delta$. Apply formulas (5) and (9) to achieve

$$\alpha \geq (a|c)_{1_G} \geq \min\{|a|_G - \alpha, |c|_G - \alpha, (ab|cd)_{1_G}\} - 2\delta.$$

Observing $|a|_G - \alpha \geq |g_{i_k}^n|_G - |x_{i_k}|_G - \alpha > \alpha + 2\delta$ and $|c|_G - \alpha > \alpha + 2\delta$, we can conclude that

$$(y_{k-1}|y_{k+1})_{y_k} = (ab|cd)_{1_G} \leq \alpha + 2\delta < C_0.$$

At last, let us estimate the product $(y_{l-1}|y_{l+1})_{y_l} = \left((x_{i_l} g_{i_l}^n)^{-\epsilon_k} h_{l-1}^{-1} \middle| h_l\right)_{1_G}$. Denote $a = (x_{i_l} g_{i_l}^n)^{-\epsilon_k}$, $b = h_{l-1}^{-1}$ and $c = h_l$.

Using formula (3) and the definition of α one can obtain

$$\alpha \geq (a|c)_{1_G} \geq \min\left\{|x_{i_l} g_{i_l}^n|_G - \alpha, (ab|c)_{1_G}\right\} - \delta.$$

As before, the latter implies that

$$(y_{l-1}|y_{l+1})_{y_l} = (ab|c)_{1_G} \leq \alpha + \delta < C_0.$$

Thus, we have shown that

$$(y_{k-1}|y_{k+1})_{y_k} < C_0 \text{ for each } k = 1, 2, \ldots, l. \tag{10}$$

Now we need to find a lower bound for the lengths of the sides in the broken line $[y_0, y_1, \ldots, y_{l+1}]$.

Let $0 \leq k \leq l-1$. Note that $|ab|_G \geq |b|_G - (a^{-1}|b)_{1_G}$ for any $a, b \in G$. Hence

$$\|[y_k, y_{k+1}]\| = |h_k (x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}|_G \geq \\ |(x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}|_G - \left(h_k^{-1} \middle| (x_{i_{k+1}} g_{i_{k+1}}^n)^{\epsilon_{k+1}}\right)_{1_G} \geq |g_{i_{k+1}}^n|_G - |x_{i_{k+1}}|_G - \alpha.$$

Applying inequality (8) we obtain

$$\|[y_k, y_{k+1}]\| > 2C_1 \quad \text{if } 0 \leq k \leq l-1. \tag{11}$$

Depending on the length of the last side, $\|[y_l, y_{l+1}]\| = |h_l|_G$, there can occur two different situations.

Case 1: $\|[y_l, y_{l+1}]\| = |h_l|_G \leq C_1$. Then we can use inequalities (10) and (11) to apply Lemma 4.3 to the geodesic broken line $p' = [y_0, \ldots, y_l]$. Hence $[y_0, y_l] \subset \mathcal{O}_{14\delta}(p')$ and $d(y_0, y_l) \geq \|p'\|/2$.

Since geodesic triangles in $\Gamma(G, \mathcal{A})$ are δ-slim, we have

$$[y_0, y_{l+1}] \subset \mathcal{O}_\delta\big([y_0, y_l] \cup [y_l, y_{l+1}]\big) \subset \mathcal{O}_{\delta + C_1}\big([y_0, y_l]\big) \subset \mathcal{O}_{15\delta + C_1}(p').$$

Now, if $l \geq 1$ in the presentation (6), one can use (11) to obtain

$$d(y_0, y_{l+1}) \geq d(y_0, y_l) - d(y_l, y_{l+1}) \geq \|p'\|/2 - C_1 \geq \|[y_0, y_1]\|/2 - C_1 > 0.$$

Case 2: $\|[y_l, y_{l+1}]\| = |h_l|_G > C_1$. Then we can apply Lemma 4.3 to the broken line $p = [y_0, \ldots, y_l, y_{l+1}]$, thus achieving

$$[y_0, y_{l+1}] \subset \mathcal{O}_{14\delta}(p).$$

As before, if $l \geq 1$ one has

$$d(y_0, y_{l+1}) \geq \|p\|/2 > 0.$$

Thus, in either case we have established the following properties:

$$[y_0, y_{l+1}] \subset \mathcal{O}_{15\delta+C_1}(p) \quad \text{and} \tag{12}$$

$$d(y_0, y_{l+1}) > 0. \tag{13}$$

The inequality (13) implies that $w \neq 1_G$ in the group G for any element $w \in M$ having a "reduced" presentation (6) with $l \geq 1$. Therefore

$$M \cong H * \langle x_1 g_1^n \rangle * \cdots * \langle x_s g_s^n \rangle.$$

As $n \geq N$ is fixed, one is able to define the constants

$$\zeta = \max_{1 \leq i \leq s} \{|x_i g_i^n|_G\} < \infty \quad \text{and} \quad \varepsilon = 16\delta + C_1 + \eta + \zeta.$$

We will finish the proof by showing that $[y_0, y_{l+1}] \subset \mathcal{O}_\varepsilon(M)$ which implies that M is ε-quasiconvex.

Consider an arbitrary $k \in \{0, 1, \ldots, l\}$. Using the construction of $y_k \in M$ and ζ, and δ-hyperbolicity of the Cayley graph we achieve

$$[y_k, y_{k+1}] \subset \mathcal{O}_{\delta+\zeta}([y_k, y_k h_k]). \tag{14}$$

H is η-quasiconvex, therefore $[1_G, h_k] \subset \mathcal{O}_\eta(H)$. The metric in $\Gamma(G, \mathcal{A})$ is invariant under the action of G by left translations, consequently

$$[y_k, y_k h_k] \subset \mathcal{O}_\eta(y_k H) \subset \mathcal{O}_\eta(M).$$

Combining the latter formula with (14) leads us to

$$[y_k, y_{k+1}] \subset \mathcal{O}_{\delta+\eta+\zeta}(M) \quad \text{for each } k.$$

Finally, an application of (12) yields

$$[y_0, y_{l+1}] \subset \mathcal{O}_{15\delta+C_1+\delta+\eta+\zeta}(M) = \mathcal{O}_\varepsilon(M),$$

as desired. □

5. Hyperbolic groups with engulfing

Proof of Theorem 1. Since every elementary group is residually finite, it is sufficient to consider the case when G is non-elementary. Let $x_1, \ldots, x_s$ be the generators of G.

Define $K = \bigcap_{L \leq G, |G:L| < \infty} L$; then K is normal in G. Suppose K is infinite. If the subgroup K were elementary, then it would be quasiconvex (this follows directly from Lemmas 3.4 and 2.1). Hence, according to a result proved by Mihalik and Towle [9] (see also [14, Cor. 2]), it would have a finite index in G, thus the group G would also have to be elementary. Therefore K can not be elementary.

Now we can apply Lemma 3.5 to find pairwise non-commensurable K-suitable elements $g_1, \dots, g_{s+1} \in K$. Since the trivial subgroup $H = \{1_G\} \le G$ is quasiconvex, we can use Theorem 5 to show that the subgroups $M = \langle x_1 g_1^n, \dots, x_s g_s^n \rangle$ and $M' = \langle x_1 g_1^n, \dots, x_s g_s^n, g_{s+1}^n \rangle$ are free (of ranks s and $(s+1)$ respectively) and quasiconvex in G for some sufficiently large $n \in \mathbb{N}$.

Note that M is a proper (infinite index) subgroup of G because $|M' : M| = \infty$. According to our assumptions, there exists a proper finite index subgroup L in G with $M \le L$. By the construction, $K \le L$, thus $x_i g_i^n, g_i \in L$ for each $i = 1, \dots, s$. Consequently $x_i \in L$ for each $i = 1, \dots, s$, contradicting with properness of L.

Therefore K is finite. If G is torsion-free then K is trivial, and, thus, G is residually finite. □

As $E(G)$ is the maximal finite normal subgroup in G, we obtain the following statement right away:

Corollary 3. *With the assumptions of Theorem 1, suppose, in addition, that* $E(G) = \{1_G\}$. *Then* G *is residually finite.*

Below it will be convenient to use the following equivalence relation between subsets of a group G defined in [14]: for any $A, B \subseteq G$ such that $A \preceq B$ and $B \preceq A$ we will write $A \approx B$.

Remark 5.1. ([14, Remark 3]) If $A, B \subseteq G$, $A \approx B$ and A is quasiconvex, then B is also quasiconvex.

In particular, if $A \le B$ are subgroups of G and A has finite index in B then $A \approx B$. Hence A is quasiconvex if and only if B is quasiconvex.

Lemma 5.1. *Let* G *be a residually finite hyperbolic group and let* $H \le G$ *be a quasiconvex subgroup. Suppose that every proper quasiconvex subgroup of* G *is engulfed. Then* H *has a finite index in its profinite closure* K *in* G.

Proof. Arguing by the contrary, assume that $|K : H| = \infty$. Since the group G is residually finite, any finite subset is closed in the profinite topology. Thus H is infinite; hence K is non-elementary.

Choose some generating set $x_1, \dots, x_s$ of the group G. Since G is residually finite, it has a finite index subgroup G_1 satisfying

$$G_1 \cap \left(E(K) \cup \bigcup_{i=1}^{s} x_i E(K) x_i^{-1} \right) = \{1_G\}. \tag{15}$$

Define $H_1 = H \cap G_1$; then $|H : H_1| < \infty$ and, according to Remark 5.1, H_1 is quasiconvex. The profinite closure K_1 of H_1 in G has a finite index in K, therefore K_1 is non-elementary and $|K_1 : H_1| = \infty$. The definition of a finite index subgroup implies that there is $l \in \mathbb{N}$ such that $g^l \in K_1$ for each $g \in K$. Since $E(g) = E(g^l)$ for any $g \in K^0$ we have

$$E(K) \subseteq E(K_1) = \bigcap_{g \in (K_1)^0} E(g) \subseteq \bigcap_{g \in K^0} E(g^l) = \bigcap_{g \in K^0} E(g) = E(K).$$

Thus $E(K_1) = E(K)$.

By Lemma 3.3, $|K_1 : (K_1 \cap H_1^g)| = \infty$ for every $g \in G$ and we can use Corollary 2 to obtain a free subgroup $F \le K_1$ of rank 2 satisfying $F \cap H_1^G = \{1_G\}$ and $E(F) = E(K_1) = E(K)$.

According to Lemma 3.5 there exist elements $g_1, \dots, g_{s+1} \in F$ which are pairwise non-commensurable and F-suitable. Consequently $E(g_i) = \langle g_i \rangle_\infty \times E(K)$ and, since $E(K)$ is finite, formula (15) implies that

$$E(g_i) \cap H_1 = E(K) \cap H_1 = \{1_G\},\ i = 1, 2, \dots, s+1, \quad \text{and}$$
$$E(g_i) \cap x_i^{-1} H_1 x_i = E(K) \cap x_i^{-1} H_1 x_i = \{1_G\},\ i = 1, 2, \dots, s.$$

Now we apply Theorem 5 to find $n \in \mathbb{N}$ such that the subgroups

$$M = \langle H_1, x_1 g_1^n, \dots, x_s g_s^n \rangle \quad \text{and} \quad M' = \langle H_1, x_1 g_1^n, \dots, x_s g_s^n, g_{s+1}^n \rangle$$

are quasiconvex in G and $M' = M * \langle g_{s+1}^n \rangle_\infty \le G$. Thus, $|M' : M| = \infty$ and M is a proper subgroup of G.

The subgroup M is engulfed by our assumptions, therefore G has a proper finite index subgroup L containing M. Observe that $K_1 \le L$ because $H_1 \le M \le L$, and, since $x_i g_i^n, g_i \in M \cup K_1 \subset L$, we get $x_i \in L$ for each $i = 1, \dots, s$. The latter implies $G = L$ – a contradiction. $\square$

We will now prove Theorem 3 which strengthens the the statement of the previous lemma.

Proof of Theorem 3. We can assume that G is non-elementary because any elementary group is LERF. Since G is residually finite, there is a finite index subgroup $G_1 \le G$ with $G_1 \cap E(G) = \{1_G\}$.

Take an arbitrary quasiconvex subgroup $H \le G$ and set $H_1 = H \cap G_1$. As it follows from Remark 5.1, H_1 is quasiconvex in G. Therefore, according to Lemma 5.1, H_1 has a finite index in its profinite closure K_1 in G.

If $H_1 = K_1$, i.e., H_1 is closed in the profinite topology on G, then so is H. Thus, we can suppose that $H_1 \neq K_1$. Consequently $|G : H_1| = \infty$, and thus $|G : K_1| = \infty$.

The subgroup K_1 is quasiconvex according to Remark 5.1, hence we can apply Lemma 3.8 to find a G-suitable element $g \in G$ such that $\langle g \rangle_\infty \cap K_1 = \{1_G\}$. Since $E(g) = \langle g \rangle \times E(G)$, $E(G)$ is finite and $K_1 \le G_1$, we have

$$E(g) \cap K_1 = E(G) \cap K_1 = \{1_G\}.$$

Now we can apply Theorem 5 to find a number $n \in \mathbb{N}$ such that the subgroups $M = \langle H_1, g^n \rangle$ and $M' = \langle K_1, g^n \rangle$ are quasiconvex in G and $M' \cong K_1 * \langle g^n \rangle_\infty$.

Using properties of free products, we observe that $M \le M'$ and $|M' : M| = \infty$ because $H_1 \lneqq K_1$. On the other hand, M' is contained inside of the profinite closure of M in G. Thus we achieve a contradiction with the claim of Lemma 5.1. $\square$

Before proceeding with the next statement, we need to recall some facts concerning quasiisometries of metric spaces. Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces with metrics $d(\cdot,\cdot)$ and $e(\cdot,\cdot)$ respectively. A map $f : \mathcal{X} \to \mathcal{Y}$ is called a *quasiisometry* if there are constants $D_1 > 0$ and $D_2 \ge 0$ such that

$$D_1^{-1} d(a,b) - D_2 \le e(f(a), f(b)) \le D_1 d(a,b) + D_2 \quad \forall\ a, b \in \mathcal{X}.$$

The spaces $\mathcal{X}$ and $\mathcal{Y}$ are said to be *quasiisometric* if there exists a quasiisometry $f : \mathcal{X} \to \mathcal{Y}$ whose image is *quasidense* in $\mathcal{Y}$, i.e., there exists $\varepsilon \ge 0$ such that for each $y \in \mathcal{Y}$ there is $x \in \mathcal{X}$ with $e(y, f(x)) \le \varepsilon$.

M. Gromov [5] showed that if $\mathcal{X}$ is hyperbolic and quasiisometric to $\mathcal{Y}$ (through some map $f : \mathcal{X} \to \mathcal{Y}$) then the space $\mathcal{Y}$ is hyperbolic too. He also noted that in this case the image $f(Q)$ of any quasiconvex subset $Q \subseteq \mathcal{X}$ will be quasiconvex in $\mathcal{Y}$.

Proof of Theorem 2. Note that Q is a finite normal subgroup of G by Theorem 1.

Consider the quotient $G_1 = G/Q$ together with the natural homomorphism $\psi : G \to G_1$. Since Q is finite, ψ is a quasiisometry between G and G_1 (G_1 is equipped with the word metric induced by the image of the finite generating set of G). Therefore, G_1 is also hyperbolic and any preimage map $\bar{\psi}^{-1} : G_1 \to G$ (which maps an element of G_1 to some element of G belonging to the corresponding left coset modulo Q) is a quasiisometry as well.

Choose an arbitrary proper quasiconvex subgroup $H_1 \le G_1$. Then $\bar{\psi}^{-1}(H_1)$ is a quasiconvex subset of G and

$$\bar{\psi}^{-1}(H_1) \subseteq \psi^{-1}(H_1) \subseteq \bar{\psi}^{-1}(H_1) \cdot Q,$$

where $\psi^{-1}(H_1)$ is the full preimage of H_1 in G.

As Q is finite, the above formula implies $\bar{\psi}^{-1}(H_1) \approx \psi^{-1}(H_1)$. Therefore $\psi^{-1}(H_1)$ is quasiconvex in G by Remark 5.1. According to our assumptions, there is a proper finite index subgroup $L \le G$ containing $\psi^{-1}(H_1)$. By definition, $Q \le L$, hence $\psi(L)$ is a proper finite index subgroup of G_1 with $H_1 \le \psi(L)$.

Thus, we have shown that G_1 also engulfs each proper quasiconvex subgroup. By the construction, G_1 is residually finite and, therefore, GFERF (Theorem 3).

Consider any quasiconvex subgroup $H \le G$. Then $\psi(H)$ is quasiconvex in G_1 and, thus, it is closed in the profinite topology on G_1. The homomorphism ψ is a continuous map if G and G_1 are equipped with their profinite topologies, thus the full preimage $\psi^{-1}(\psi(H)) = H \cdot Q$ is closed in G. Obviously $H, Q \le K$ (where K is the profinite closure of H in G), hence $K = HQ$. Q.e.d. □

6. Free products of GFERF groups

In the previous section we considered hyperbolic groups which engulf every proper quasiconvex subgroup. Let us name them *QE-groups*, for brevity.

As it can be seen from Theorem 2, any QE-group G is very close to being GFERF. In fact, G is quasiisometric to the quotient $G/E(G)$ which is GFERF by Corollary 3 and Theorem 3. Nevertheless, the answer to the question whether each QE-group is GFERF is still unclear. Theorem 2 would yield a positive answer if a free product of any two QE-groups were a QE-group itself. Unfortunately, the author is unable to prove this; actually, he doubts if this is true in general.

However, the following statement, proved by R. Burns, can be used to show that a free product of GFERF-groups is, again, GFERF:

Lemma 6.1. ([2, Thm. 1.1]) *Suppose G is a free product of its subgroups G_i indexed by some set I, and let H be a finitely generated subgroup. If for each $i \in I$, $g \in G$, the subgroup $(H^g \cap G_i)$ is G_i-separable, then H is G-separable.*

Let H be a subgroup of a group G. Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite generating sets for G and H respectively and $|\cdot|_G$, $|\cdot|_H$ are the corresponding length functions. Set $\hat{c} = \max\{|b|_G \ : \ b \in \mathcal{B}\}$. Evidently, $|h|_G \le \hat{c}|h|_H$ for all $h \in H$.

H is called *undistorted* in G if there exists a constant $c \ge 0$ such that $|h|_H \le c|h|_G$ for every $h \in H$. In a hyperbolic group G, a finitely generated subgroup is undistorted if and only if it is quasiconvex ([10, Lemma 1.6]).

Proof of Theorem 4. It is well known that a free product of hyperbolic groups is a hyperbolic group (see, for instance, [4, 1.34]). Thus, G is hyperbolic. Clearly, the subgroups G_1 and G_2 are undistorted in G; consequently, they are quasiconvex.

Choose an arbitrary quasiconvex subgroup $H \le G$, an element $g \in G$ and $i \in \{1,2\}$. The subgroup H^g is quasiconvex by Remark 2.1. Since the intersection of two quasiconvex subgroups is quasiconvex ([21, Prop. 3]), $(H^g \cap G_i)$ is quasiconvex in G. Consequently, $(H^g \cap G_i)$ is undistorted in G, and, hence, it is undistorted in G_i. Thus $(H^g \cap G_i)$ is G_i-separable because G_i is GFERF.

According to Lemma 6.1, H is G-separable. Q.e.d. □

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